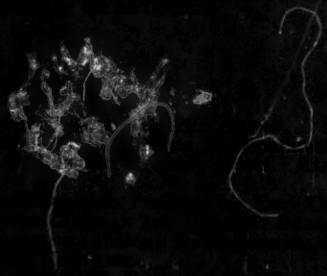

THE CRAMÉR-SMIRNOV TEST IN THE PARAMETRIC CASE

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Summary. The "goodness of fit" problem, consisting of comparing the empirical and hypothetical cumulative distribution functions (cdf's), is treated here for the case when an auxiliary parameter is to be estimated. This extends the Cramér-Smirnov and von Mises test to the parametric case, a suggestion of Cramér [1], see also [2]. The characteristic function of the limiting distribution of the test function is found by consideration of a Guassian stochastic process.

1. Introduction; position of the problem. Let $X_1, X_2, \cdots, X_n$ be independent observations (random variables) coming from a population whose absolutely continuous cdf is $G(x)$. Let I be a nondegenerate interval on the real axis and suppose, for each ξ contained in the interior of I, that $F(x; \xi)$ is a cdf. In this paper we treat the problem of testing the hypothesis H,

$$H \quad : \quad G(x) = F(x; \xi) \qquad \textit{for some unspecified} \quad \xi \, \varepsilon \, I. \tag{1.1}$$

About the only test of H at present available seems to be the usual χ^2 test, discussed recently by Cochran [3].

For a somewhat different hypothesis H_0,

$$H_0 \quad : \quad G(x) = F(x) = F(x; \xi_0) \qquad \textit{for a specified} \quad \xi_0 \, \varepsilon \, I, \tag{1.2}$$

the following test of Cramér [2], Smirnov [4], and von Mises [5] is available. Let $F_n(x)$ be the empirical cdf of the data; that is, $F_n(x) = k/n$ if k of the X_i, with $i = 1, 2, \cdots, n$, are less than x, for $-\infty < x < \infty$. The test function is then

$$W_n^2 = n \int_{-\infty}^{\infty} [F_n(x) - F(x; \xi_0)]^2 \, dF(x; \xi_0), \tag{1.3}$$

and H_0 is rejected if W_n^2 is suitably large. The limiting distribution of W_n^2, if H_0 is true, was given by Smirnov; it has been tabulated recently [6].

This test of H_0 has certain attractive properties not possessed by the usual χ^2 test. It does not require a subjective grouping of the data into classes, it is distribution free for all n (if H_0 is true), and it is consistent (i.e., has limiting power 1). Appraisals of the χ^2 test and this test are given by Cochran [3] and Birnbaum [7].

In an effort to modify the W_n^2 test to treat the hypothesis H of (1.1), we shall consider, following a suggestion of Cramér, the test function

$$C_n^2 = n \int_{-\infty}^{\infty} [F_n(x) - F(x; \hat{\theta}_n)]^2 \, dF(x; \hat{\theta}_n), \tag{1.4}$$

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where $\hat{\theta}_n$ is a suitable estimator for the unknown parameter ξ in $F(x; \xi)$, and is a function of $X_1, X_2, \cdots, X_n$. The hypothesis H is to be rejected if C_n^2 is sufficiently large. If $X_1', X_2', \cdots, X_n'$ is a rearrangement of the sample $X_1, X_2, \cdots, X_n$ so that $X_1' < X_2' < \cdots < X_n'$, for computational purposes C_n^2 can be expressed more simply as

$$C_n^2 = \frac{1}{12n} + \sum_{j=1}^{n} \left[F(X_j'; \hat{\theta}_n) - \frac{2j-1}{2n} \right]^2. \tag{1.5}$$

The main analytical task is to calculate the distribution of C_n^2 when H is true; it is accomplished in the present paper. It turns out that matters depend crucially on the estimator $\hat{\theta}_n$ chosen. There are two essentially distinct cases.

If it happens that, among other conditions stated in Theorem 2.1, $\hat{\theta}_n$ is an estimator such that $nE\{(\hat{\theta}_n - \theta)^2\} \to 0$, where θ is the true value of ξ, that is, $G(x) = F(x; \theta)$, then the limiting distribution of C_n^2 is the same as the limiting distribution of W_n^2 given by (1.3). This is known and tabulated.

In the more general case when θ does not admit this so-called superefficient estimate but $\sqrt{n}(\hat{\theta}_n - \theta)$ has a limiting Gaussian distribution with mean 0 and variance $\sigma^2 > 0$, then the limiting distributions of C_n^2 and W_n^2 are not the same; the limiting distribution of C_n^2 is that of $\int_0^1 Y^2(t)\, dt$, where $Y(t)$ is a certain Gaussian stochastic process. The process has a specially simple structure if θ admits a minimum variance (or efficient) estimator $\hat{\theta}_n$, or if the maximum likelihood estimator is asymptotically efficient in the sense of Cramér [1].

In this latter case, we determine the ch.fn. of the limiting distribution of C_n^2 explicitly. We give several illustrations, but unfortunately even in simple, important cases (such as, a normal distribution with an unknown mean) the resulting ch.fn. appears very difficult to invert.

Unlike the W_n^2 test, the C_n^2 test is not distribution free. In general it depends on the unknown true value of ξ, though in important special cases (including the case when ξ is a scale or location parameter) the asymptotic test depends only on the structure of the family $F(x; \xi)$ and not upon the particular value of ξ obtaining; that is, the test is parameter-free, so to speak.

2. The superefficient case. Suppose that H as given by (1.1) is true, and let the true unknown value of the parameter ξ be θ, with θ an interior point of I. Let the density corresponding to $F(x; \xi)$ be $f(x; \xi)$. Denote as before

$$n \int_{-\infty}^{\infty} (F_n(x) - F(x; \theta))^2 \, dF(x; \theta) = W_n^2.$$

THEOREM 2.1. *Assume that the estimator $\hat{\theta}_n$ and the distribution $F(x; \xi)$ have the following properties:*

1) $nE\{(\hat{\theta}_n - \theta)^2\} \to 0$, $n \to \infty$;

2) *For $\xi, \xi' \in I$, $F(x; \xi)$ satisfies a Lipschitz condition*

$$|F(x; \xi) - F(x; \xi')| < A(x) |\xi - \xi'|,$$

where $\Pr\{A^2(X_1) > A_0\} = 0$ *for some* $A_0 < \infty$, *the probability according to* $F(x; \theta)$.

Then $C_n^2 = W_n^2 + \delta_n$, *where* $\delta_n \to 0$ *in probability.*

PROOF. From (1.5)

$$C_n^2 = \frac{1}{12n} + \sum_{j=0}^{n} \left(F(X_j'; \hat{\theta}_n) - \frac{j - \frac{1}{2}}{n} \right)^2$$

$$= \frac{1}{12n} + \sum_{j=0}^{n} \left(\left\{ F(X_j'; \theta) - \frac{j - \frac{1}{2}}{n} \right\} + \{F(X_j'; \hat{\theta}_n) - F(X_j'; \theta)\} \right)^2$$

$$= W_n^2 + 2 \sum \left\{ F(X_j'; \theta) - \frac{j - \frac{1}{2}}{n} \right\} \{F(X_j'; \hat{\theta}_n) - F(X_j'; \theta)\}$$

$$+ \sum \{F(X_j'; \hat{\theta}_n) - F(X_j'; \theta)\}^2$$

$$= W_n^2 + 2\delta_1 + \delta_2 .$$

Then $\delta_1^2 \leqq (W_n^2 - 1/12n)\delta_2$, and

$$\delta_2 = \sum \{F(X_j'; \hat{\theta}_n) - F(X_j'; \theta)\}^2 \leqq n(\hat{\theta}_n - \theta)^2 \max_j A^2(X_j).$$

Thus $E(\delta_2) \to 0$ and $E(\delta_1^2) \leqq E\{(W_n^2 - 1/12n)^2\}^{1/2} E(\delta_2^2)^{1/2} \to 0$, and the theorem follows. A trivial consequence of the theorem is

COROLLARY 2.1. *Under the conditions* 1) *and* 2) *of Theorem* 1.1, *the limiting distribution of* C_n^2 *is the same as that of the von Mises statistic* W_n^2.

The limiting distribution of W_n^2 has been tabulated [6]; the problem in this case is solved. Two simple examples in which the estimate $\hat{\theta}_n$ is unbiased follow.

a.) It is easily verified that $\operatorname{Var}(\hat{\theta}_n) = \theta^2/n(n + 2)$, and that condition 2) of Theorem 1.1 is satisfied, when

$$I = \{\xi \mid 0 < \delta < \xi < \infty\};$$

$$\hat{\theta}_n = \frac{n + 1}{n} \max \{X_1, X_2, \cdots, X_n\}. \qquad f(x; \xi) = \begin{cases} 1/\xi, & 0 < x < \xi, \\ 0, & \text{otherwise;} \end{cases}$$

b.) Likewise, $\operatorname{Var}(\hat{\theta}_n) = k/2(n + 1)(n + 2)$, and 2) is satisfied by $f(x; \xi)$, when

$$I = \{\xi \mid -\infty < \xi < \infty\}; \qquad f(x; \xi) = \begin{cases} 1, & \xi - \frac{1}{2} < x < \xi + \frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases}$$

$$\hat{\theta}_n = \tfrac{1}{2}(U + V).$$

Here U is the kth largest X_i and V is the kth smallest X_i, for $i = 1, 2, \cdots n$, and k is independent of n (e.g., $k = 1$ gives $\hat{\theta}_n$ the midrange).

3. The regular estimation case. In general, the rather rapid decrease of $\operatorname{Var}(\hat{\theta}_n)$ to zero as expressed by condition 1) of Theorem 1.1 will not occur. In the case of regular estimation of Cramér ([1] p. 477), we will have $\operatorname{Var}(\hat{\theta}_n) \geqq A/n$ for some positive A—the Cramér-Rao inequality. But it will generally happen

that $n^{1/2-\delta}(\hat{\theta}_n - \theta)$ will converge in probability to zero for $\delta > 0$. To cover this situation, we have the following

LEMMA 3.1. *Assume that $f(x;\xi)$ and $\hat{\theta}_n$ are such that*

1) $nE\{(\hat{\theta}_n - \theta)^4\} \to 0$,

2) $\left|\frac{\partial^2 F(x;\xi)}{\partial \xi^2}\right| < g_0(x)$,

3) $\left|\frac{\partial f(x;\xi)}{\partial \xi}\right| < g_1(x)$,

for almost all x, where g_0 and g_1 are integrable from $-\infty$ to $+\infty$. The functions g_0 and g_1 and the exceptional set do not depend on ξ.

Then,

$$(3.1)\quad C_n^2 = n\int_{-\infty}^{\infty}\left\{F_n(x) - F(x) - (\hat{\theta}_n - \theta)\frac{\partial}{\partial\theta}F(x;\theta)\right\}^2 f(x;\theta)\,dx + \delta_n,$$

where $\delta_n \to 0$ in probability.

For almost all x we have

$$F(x;\hat{\theta}_n) = F(x;\theta) + (\hat{\theta}_n - \theta)\frac{\partial}{\partial\theta}F(x;\theta) + \tfrac{1}{2}(\hat{\theta}_n - \theta)^2 q_0 g_0(x), \qquad |q_0| < 1;$$

$$f(x;\hat{\theta}_n) = f(x;\theta) + (\hat{\theta}_n - \theta)q_1 g_1(x), \qquad |q_1| < 1.$$

Putting these expansions in expression (2.1), we obtain the lemma after some calculations.

Conditions 2) and 3) in the above lemma could be replaced by the condition, similar to that of Theorem 2.1,

$$\left|\frac{\partial FX;\xi)}{\partial\xi} - \frac{\partial F(X;\xi')}{\partial\xi'}\right| < A\,|\xi - \xi'|.$$

Thus it is necessary to study only the limiting form (if it exists) of the distribution of

$$R_n^2 = n\int_{-\infty}^{\infty}\left\{F_n(x) - F(x;\theta) - (\hat{\theta}_n - \theta)\frac{\partial}{\partial\theta}F(x;\theta)\right\}^2 f(x;\theta)\,dx.$$

A transformation which is basic in the work to follow,

$$(3.2)\qquad u = F(x;\theta),$$

defines implicitly x as a function of u. Thus $x = x(u;\theta)$ except possibly for an enumerable set of u values at which x can be defined arbitrarily, except to render it monotone nondecreasing. Next we define the function $g(u)$ as

$$(3.3)\qquad g(u) = \frac{\partial}{\partial\theta}F(x;\theta), \qquad 0 \leqq u \leqq 1,$$

and note that $g(u)$ depends in general on θ. Finally, we express the empirical cdf as a function of u. If we introduce the function $\psi_t(v)$ defined as

$$\psi_t(v) = \begin{cases} 1, & v < t, \\ 0, & v \geqq t, \end{cases}$$

then we have, with probability one,

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} \psi_x(X_j) = \frac{1}{n}\sum_{j=1}^{n} \psi_u(F(X_j\,;\theta)),$$

where u is given by (3.2.). On writing

$$Z_n(u) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} \psi_u(F(X_j\,;\theta)) - u\right), \tag{3.4}$$

$$T_n = \sqrt{n}(\hat{\theta}_n - \theta), \tag{3.5}$$

C_n^2 can be written, from (3.1), as

$$C_n^2 = \int_0^1 (Z_n(u) - T_n g(u))^2\, du + \delta_n$$

where $\delta_n \to 0$ in probability, and $g(u)$ is given by (3.3). Finally by defining the stochastic process $Y_n(u)$ as

$$Y_n(u) = Z_n(u) - T_n g(u) \tag{3.6}$$

where $Z_n(u)$ and T_n are given by (3.4) and (3.5), and $g(u)$ by (3.3), we have

$$C_n^2 = \int_0^1 Y_n^2(u)\, du + \delta_n$$

where $\delta_n \to 0$ in probability.

It follows that the limiting form of the stochastic process $Y_n(u)$ is of central importance, and we next prove

LEMMA 3.2. *Assume, in addition to conditions* 1), 2), 3) *of Lemma* 3.1, *that*

4) $nE(\hat{\theta}_n - \theta) \to 0$;

5) $T_n = \sqrt{n}(\hat{\theta}_n - \theta)$ *is a sum of independent identically distributed random variables having a limiting Gaussian distribution with variance* $\sigma^2 > 0$; *and*

6) *the conditional expectation* $nuE(\hat{\theta}_n - \theta \mid F(X_1\,;\theta) < u)$ *converges to a function* $h(u)$ *with* $0 < u < 1$ *and* $h(0) = h(1) = 0$.

Then the process $Y_n(u)$ *converges in distribution to a Gaussian process* $Y(u)$ *with mean* 0 *and covariance*

$$\begin{aligned} \rho(u, v) &= E(Y(u)Y(v)) \\ &= \min(u, v) - uv - g(u)h(v) - g(v)h(u) + \sigma^2 g(u)g(v). \end{aligned} \tag{3.7}$$

The expression "$Y_n(u)$ converges to $Y(u)$ in distribution" means that for every finite set $u_1, u_2, \cdots, u_k$ the joint cdf of $Y_n(u_1), Y_n(u_2), \cdots, Y_n(u_k)$ converges to the joint cdf of $Y(u_1), Y(u_2), \cdots, Y(u_k)$. The proof of the lemma

is quite straightforward. The process $Z_n(u)$ defined by (3.4) is known to converge in distribution to a Gaussian process with mean 0 and covariance $k(u, v) = E(Z_n(u)Z_n(v))$ given by

$$k(u, v) = \min(u, v) - uv, \tag{3.8}$$

with $k(u, v)$ being independent of n (see, for example, [6]). By 5) T_n has a limiting Gaussian distribution with mean 0 and variance σ^2. It follows from the multidimensional central limit theorem that the linear combination $Y_n(u) = Z_n(u) - T_n g(u)$ converges in distribution to a Gaussian process whose mean is 0. Thus will be sufficient to verify that the limiting covariance $\rho_n(u, v) = E(Y_n(u)Y_n(v))$ converges to $\rho(u, v)$ given by (3.7).

Denoting by $h_n(u)$ the function

$$h_n(u) = E(Z_n(u)T_n) \tag{3.9}$$

and using (3.4) and (3.5), we have, by condition 5),

$$\begin{aligned} h_n(u) &= \sqrt{n}E\{[(1/n)\sum \psi_u(F(X_j\,;\,\theta)) - u][\sqrt{n}(\hat{\theta}_n - \theta)]\} \\ &= \sum E[\psi_u(F(X_j\,;\,\theta))(\hat{\theta}_n - \theta)] - nuE(\hat{\theta}_n - \theta) \\ &= nE\{(\hat{\theta}_n - \theta)\psi_u(F(X_1\,;\,\theta))\} - nuE(\hat{\theta}_n - \theta) \\ &= nuE\{\hat{\theta}_n - \theta \mid F(X_1\,;\,\theta) < u\} - nuE(\hat{\theta}_n - \theta). \end{aligned} \tag{3.10}$$

Consequently by 4) and 6), $h_n(u) \to h(u)$ for $0 \leqq u \leqq 1$. The covariance $\rho_n(u, v)$ is

$$\begin{aligned} \rho_n(u, v) &= E\{[Z_n(u) - T_n g(u)][(Z_n(v) - T_n g(v)]\} \\ &= k(u, v) - h_n(u)g(v) - h_n(v)g(u) + \sigma_n^2 g(u)g(v), \end{aligned} \tag{3.11}$$

where $\sigma_n^2 = \mathrm{Var}(T_n) \to \sigma^2$ and $k(u, v)$ is given by (3.8). Thus we obtain $\rho_n(u, v) \to \rho(u, v)$ for $\rho(u, v)$ as in (3.7), and the lemma is established.

It might be concluded that the limiting distribution of C_n^2 is the distribution of $C^2 = \int_0^1 Y^2(u)\,du$ where $Y(u)$ is a Gaussian process with mean 0 whose covariance is given by (3.7), following Doob's [8] heuristic approach. But we shall prove this fact in the next section only when the estimator $\hat{\theta}_n$ is further specialized.

Two further properties of the function $h_n(u)$ conclude this section.

LEMMA 3.3. *For the function $h_n(u)$ as defined by* (3.9),

1) $|h_n(u)| \leqq \sigma_n\sqrt{u(1-u)}$, $\qquad 0 \leqq u \leqq 1$;

2) $h_n'(u) = nE\{\hat{\theta}_n - \theta \mid F(X_1\,;\,\theta) = u\} - nE(\hat{\theta}_n - \theta)$.

The first relation follows immediately by Schwarz' inequality from (3.9) by using (3.8) with $u = v$ and $\mathrm{Var}(T_n) = \sigma_n^2$. For the second relation we use (3.10) to give

$$h_n(u + \delta) - h_n(u) = n\delta E\{(\hat{\theta}_n - \theta) \mid [u < F(X_1\,;\,\theta) \leqq u + \delta]\} - n\delta E\{\hat{\theta}_n - \theta\}.$$

By letting $\delta \to 0$, 2) follows.

For $n \to \infty$, $h_n(u)$ converges to $h(u)$ and σ_n^2 converges to σ^2. Thus we have $|h(u)| \leqq \sigma\sqrt{u(1-u)}$. However, we cannot conclude directly that $\lim h_n'(u)$ exists.

4. Case of an efficient estimator. As yet, we have given no special attention to the choice of the estimator $\hat{\theta}_n$. It might be thought that, paralleling the principle of minimum χ^2, we should choose $\hat{\theta}_n$ so as to make C_n^2, as given by (2.1), a minimum. However, as is often the case with minimum χ^2, this does not lead to usable results. However, precisely as in the χ^2 case, the maximum likelihood principle does lead to a certain ideal properties for C_n^2, at least asymptotically.

In this section, we assume that Cramér's conditions ([1], pp. 477–489) for a regular, unbiased efficient (or minimum variance) estimate are satisfied. Following Cramér, we simply term the estimate efficient. Then all the conditions 1) through 6) of Lemma 3.2 are satisfied, as noted below, with the possible exception of condition 2), which we shall further presume satisfied. Condition 2) is also postulated by Cramér for the maximum likelihood case which we consider in the next section. The efficient estimator is unbiased so that condition 4) is satisfied, and implies besides that the likelihood function

$$L = \prod_{j=1}^{n} f(X_j; \xi)$$

has the property that if $\xi = \hat{\theta}_n$ is a root of $(\partial/\partial\xi) \log L = 0$, then, defining

$$\sigma^2 = 1/E\left\{\left[\frac{\partial}{\partial\theta} \log f(X_1; \theta)\right]^2\right\} = \left(\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta} \log f(x; \theta)\right]^2 f(x; \theta)\, dx\right)^{-1}, \tag{4.1}$$

we have

$$\frac{\partial}{\partial\xi} \log L = \sum_{j=1}^{n} \frac{\partial}{\partial\xi} \log f(X_j; \xi) = \frac{n}{\sigma^2}(\hat{\theta}_n - \xi).$$

By putting $\xi = \theta$, this yields

$$\frac{n}{\sigma^2}(\hat{\theta}_n - \theta) = \sum_{j=1}^{n} \frac{\partial}{\partial\theta} \log f(X_j; \theta) = \frac{\sqrt{n}}{\sigma^2} T_n. \tag{4.2}$$

It then follows that the variance of T_n is σ^2, independent of n, given by (4.1), and that conditions 5) and 6) are satisfied.

For condition 7) of Lemma 3.2, we multiply through the last equality by σ^2 and take conditional expectations of both sides under the condition that $F(X_1; \theta) = u$. Then on using 2) of Lemma 3.3, and the fact that $E[(\partial/\partial\theta) \log f(X_j; \theta)] = 0$, we have

$$\begin{aligned} h_n'(u) &= \sigma^2 \sum_{j=1}^{n} E\left\{\frac{\partial}{\partial\theta} \log f(X_j; \theta) \mid F(X_1; \theta) = u\right\} \\ &= \sigma^2 E\left\{\frac{\partial}{\partial\theta} \log f(X_1; \theta) \mid F(X_1; \theta) = u\right\} \\ &= \sigma^2 \frac{\partial}{\partial\theta} \log f(x; \theta), \end{aligned}$$

using transformation (3.2). Thus $h'_n(u)$ is *independent* of n. We denote it by $h'(u)$, given by

$$h'(u) = \sigma^2 \frac{\partial}{\partial\theta} \log f(x;\theta). \tag{4.3}$$

Because of this simple formula for $h(u)$, making it proportional to $g(u)$ as we see immediately, $\rho(u, v)$ simplifies and renders the process $Y(u)$ of Lemma 3.2 manageable.

From the definition of $g(u)$ in (3.3) we have immediately $g(0) = g(1) = 0$ and

$$g'(u) = \frac{\partial}{\partial\theta} f(x;\theta) \frac{dx}{du} = \frac{1}{f(x;\theta)} \frac{\partial}{\partial\theta} f(x;\theta) = \frac{\partial}{\partial\theta} \log f(x;\theta) = \frac{1}{\sigma^2} h'(u).$$

Since $h(0) = 0$ by 1) of Lemma 3.3, by integrating we obtain $h(u) = \sigma^2 g(u)$. Thus it follows that we can define a function $\varphi(u)$,

$$\varphi(u) = \frac{h(u)}{\sigma} = \sigma g(u), \tag{4.4}$$

where $h'(u)$ is given by (4.3) and $g(u)$ by (3.3), σ being given by (4.1) and x by (3.2).

Putting these values for $h(u)$ and $g(u)$ in (3.11), we obtain

LEMMA 4.1. *In the case of an efficient estimator, the process $Y_n(u)$ given by* (3.6) *has mean* 0 *and covariance*

$$\rho(u, v) = k(u, v) - \varphi(u)\varphi(v), \tag{4.5}$$

independent of n, where $k(u, v)$ is given by (3.8) *and $\varphi(u)$ by* (4.4). *The function $\varphi(u)$ has the properties*

$$\text{a)} \quad |\varphi(u)| \leqq \sqrt{u(1-u)}, \qquad \text{b)} \quad \int_0^1 \varphi'^2(u)\, du = 1.$$

Condition a) follows from 1) of Lemma 3.3. To prove b), from (4.4) and (4.3) we have

$$\int_0^1 \varphi'^2(u)\, du = \frac{1}{\sigma^2} \int_0^1 h'^2(u)\, du = \sigma^2 \int_0^1 \left[\frac{\partial}{\partial\theta} \log f(x;\theta)\right]^2 du$$

$$= \sigma^2 \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial\theta} \log f(x;\theta)\right]^2 f(x;\theta)\, dx = 1,$$

when we use (4.1).

Thus we have shown that, when an efficient estimator $\hat{\theta}_n$ exists, the limiting distribution of C_n^2 is the same as the limiting distribution of $\int_0^1 Y_n^2(u)\, du$, where $Y_n(u)$ has mean 0 and covariance $\rho(s, t)$ given by (4.5), and approaches a Gaussian process $Y(u)$ in distribution. We need to show that this limiting distribution is the same as the distribution of $C^2 = \int_0^1 Y^2(u)\, du$ where $Y(u)$ is a Gaussian

process with mean 0 and covariance $\rho(s, t)$. To state the theorem formally,

THEOREM 4.1. *In the case of an efficient estimator,*

$$\lim_{n\to\infty} \Pr\{C_n^2 < x\} = \Pr\left\{\int_0^1 Y^2(u)\,du < x\right\},$$

where $Y(u)$ is a Gaussian process with mean 0 and covariance $\rho(u, v)$ given by (4.5).

Actually we are going to suppose that $\varphi(u)$ satisfies the

AUXILIARY ASSUMPTION. $\varphi''(u)$ exists almost everywhere for $0 \leqq u \leqq 1$ and

$$\int_0^1 |\varphi''(u)|\, u(1-u) \log\log \frac{1}{u(1-u)}\,du < \infty. \tag{4.6}$$

We first show the $Y_n(u)$ process can be expressed in terms of $Z_n(u)$ and that we have the representation

$$Y_n(u) = Z_n(u) + \varphi(u)\int_0^1 \varphi''(t)Z_n(t)\,dt. \tag{4.7}$$

From (3.6) and (4.2) we have merely to show

$$\int_0^1 \varphi''(t)Z_n(t)\,dt = -\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{n}\frac{\partial}{\partial\theta}\log f(X_j;\theta).$$

By using (3.4) for $Z_n(t)$, integrating by parts, and using (4.3) to obtain $\varphi'(t) = \sigma(\partial/\partial\theta)\log f(x;\theta)$, where $t = F(x;\theta)$, the result follows.

Now the process $Z(u)$, which is the limit in distribution of $Z_n(u)$ given by (3.4), is Gaussian with mean 0 and, by (3.8), covariance $k(s, t) = \min(s, t) - st$. It has been shown [6] that the random variable $\int_0^1 \varphi''(u)Z(u)\,du$ exists (and is Gaussian with mean 0) when the auxiliary assumption (4.6) is satisfied. Then the process

$$Y(u) = Z(u) + \varphi(u)\int_0^1 \varphi''(t)Z(t)\,dt \tag{4.8}$$

is Gaussian with mean 0 and covariance

$$E(Y(u)Y(v)) = k(u, v) + \varphi(u)\int_0^1 \varphi''(t)k(v, t)\,dt + \varphi(v)\int_0^1 \varphi''(t)k(t, u)\,dt + \varphi(u)\varphi(v)\int_0^1\int_0^1 \varphi''(t)\varphi''(s)k(t, s)\,dt\,ds.$$

Since $\int_0^1 \varphi''(t)k(v, t)\,dt = -\varphi(v)$, the covariance becomes

$$\begin{aligned} E(Y(u)Y(v)) &= k(u, v) - 2\varphi(u)\varphi(v) + \varphi(u)\varphi(v)\int_0^1 \varphi''(t)\varphi(t)\,dt \\ &= k(u, v) - 2\varphi(u)\varphi(v) + \varphi(u)\varphi(v)\int_0^1 \varphi'^2(t)\,dt \\ &= k(u, v) - \varphi(u)\varphi(v) = \rho(u, v), \end{aligned}$$

where we have used b) of Lemma 4.1.

Hence (4.8) is a *representation* of the process $Y(u)$ of Theorem 4.1 in terms of the process $Z(u)$. Then since $\int_0^1 Y_n^2(u)\,du$ is a functional of $Z_n(u)$ continuous in the uniform topology by (4.7), it follows from a theorem of Donsker [9] that its limiting distribution is the same as the distribution of $\int_0^1 Y^2(u)\,du$ for the process (4.8) or the process $Y(u)$ of the theorem, and the proof is complete.

By virtue of Theorem 4.1 we can now concentrate on the process $Y(u)$ and attempt to find the distribution of $C^2 = \int_0^1 Y^2(u)\,du$.

5. Case of a maximum likelihood estimator. When an efficient estimator for θ, for finite n, does not exist, it will generally happen that a maximum likelihood estimator for θ will exist and be asymptotically efficient in the sense of Cramér. In this case, the results of Section 4 will still be valid.

We shall assume that the conditions of Cramér ([1] pp. 500–501) are satisfied. These conditions imply that conditions 1)–6) of Lemmas 3.1 and 3.2 are satisfied with the exception of condition 4). We shall suppose this condition to be satisfied also, calling it a condition of "*weak-biasedness*" on $\hat{\theta}_n$. In some cases, as with the efficient estimators, $\hat{\theta}_n$ is actually unbiased and 4) is trivially satisfied.

The following analysis parallels exactly the development in the preceding section. The formulas are all the same except for additional terms which approach 0 in the mean and with probability one. We merely indicate the development leading to theorem 5.1.

Defining σ^2 as in (4.1), we can write

$$\frac{\partial}{\partial \xi} \log L = \sum_{j=1}^{n} \frac{\partial}{\partial \xi} \log f(X_j; \xi) = \frac{n}{\sigma^2} (\hat{\theta}_n - \xi)(1 + \epsilon_n),$$

where $\epsilon_n \to 0$ in the mean and with probability one. Then (4.2) becomes, with $\epsilon_n' \to 0$ strongly,

$$\frac{n}{\sigma^2} (\hat{\theta}_n - \theta) = \sum_{j=1}^{n} \frac{\partial}{\partial \theta} \log f(X_j; \theta)(1 + \epsilon_n').$$

Consequently, $h_n'(u) \to h'(u)$ and $\sigma_n^2 = \mathrm{Var}(\sqrt{n}(\hat{\theta}_n - \theta)) \to \sigma^2$ for $n \to \infty$, where $h_n'(u)$ is given by 2) of Lemma 3.3 and $h'(u)$ by (4.3). Accordingly, $\rho_n(u, v)$ given by (3.11) converges to $\rho(u, v)$ defined by (4.5). Thus we can generalize Theorem 4.1 to

THEOREM 5.1. *If $\hat{\theta}_n$ is a weakly-biased maximum likelihood estimator satisfying Cramér's conditions, and if $\varphi(u)$ satisfies* (4.6), *then*

$$\lim \Pr\{C_n^2 < x\} = \Pr\left\{\int_0^1 Y^2(u)\,du < x\right\}$$

where $Y(u)$ is a Gaussian process with mean 0 *and covariance $\rho(s, t)$ given by* (4.5).

The theorem says that C_n^2 converges in distribution to $C^2 = \int_0^1 Y^2(u)\,du$.

6. The limiting distribution of C_n^2. In the preceding sections we have reduced the problem of finding the limiting distribution of C_n^2 under general conditions to that of finding the distribution of $\int_0^1 Y^2(u)\,du$, where $Y(u)$ is a Gaussian process with mean 0 and covariance $\rho(s, t)$ given by

$$\rho(u, v) = k(u, v) - \varphi(u)\varphi(v), \qquad k(u, v) = \min(u, v) - uv; \tag{6.1}$$

$$\varphi(u) = \sigma \frac{\partial}{\partial\theta} F(x; \theta), \qquad u = F(x; \theta);$$

$$\sigma^2 = \left[\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta} \log f(x; \theta)\right)^2 f(x; \theta)\,dx\right]^{-1}. \tag{6.2}$$

The function $\varphi(u)$ can also be determined from

$$\varphi'(u) = \sigma \frac{\partial}{\partial\theta} \log f(x; \theta), \qquad u = F(x; \theta).$$

In determining the distribution of $\int_0^1 Y^2(u)\,du$ we shall use a basic theorem due essentially to Kac-Siegert [10]. It states that the distribution of this random variable is the same as the distribution of a sum of weighted χ^2,

$$C^2 = \int_0^1 Y^2(u)\,du = \sum_{n=1}^{\infty} \frac{G_n^2}{\lambda_n},$$

where $G_1, G_2, \cdots$ are independent normally distributed random variables with mean 0 and variance 1, and $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of the kernel $\rho(s, t)$. Proof of this result for nonstationary processes follows almost immediately from [10]; it may be found, for example, in Fortet's paper [11]. We have for the ch.fn.

$$E\left\{\exp\left[it\int_0^1 Y^2(u)\,du\right]\right\} = \prod_{j=1}^{\infty}\left(1 - \frac{2it}{\lambda_j}\right)^{-1/2}.$$

This expression is merely $(D(2it))^{-1/2}$, where $D(\lambda)$ is the Fredholm determinant associated with the kernel $\rho(u, v)$ in (6.1). To state matters formally, we have

THEOREM 6.1. *Under the conditions of Theorem* 5.1,

$$\lim_{n\to\infty} E\{\exp(itC_n^2\} = [D(2it)]^{-1/2},$$

where $D(\lambda)$ *is the Fredholm determinant of the integral equation*

$$f(x) = \lambda \int_0^1 \rho(x, y)f(y)\,dy$$

in which $\rho(x, y)$ *is given by* (6.1).

Thus, for $\varphi(u)$ given, it is necessary only to determine $D(\lambda)$ and invert the ch.fn. $(D(2it))^{-1/2}$ in order to obtain the limiting distribution of C_n^2. It turns out to be possible to get a fairly explicit formula for $D(\lambda)$, but the resulting ch.fn.'s seem very difficult to invert.

We first prove a result more general than needed but which may be of independent interest.

THEOREM 6.2. *Let $k_1(x, y)$ be a symmetric, bounded positive definite kernel over the unit square $0 \leqq x, y \leqq 1$ whose Fredholm determinant $d_1(\lambda)$ has simple zeroes $0 < \lambda_1 < \lambda_2 < \cdots$, and let the corresponding eigenfunctions be $f_1(x), f_2(x), \cdots$. Let*

$$a_j = c(f_j) = \int_0^1 f_j(x)\varphi_1(x)\,dx, \qquad \varphi_1(x) \in L_2(0, 1).$$

Put $c(g) = \int_0^1 g(x)\varphi_1(x)\,dx$. Then the Fredholm determinant $D_1(\lambda)$ for the kernel

$$\rho_1(x, y) = k_1(x, y) - \varphi_1(x)\varphi_1(y)$$

is given by

$$D_1(\lambda) = d_1(\lambda)\left(1 + \lambda \sum_{n=1}^{\infty} \frac{a_n^2}{1 - \lambda/\lambda_n}\right).$$

PROOF. The integral equation

$$g(x) = \lambda \int_0^1 [k_1(x, y) - \varphi_1(x)\varphi_1(y)]\, g(y)\,dy \tag{6.3}$$

can be written

$$g(x) = -\lambda\varphi_1(x)c(g) + \lambda \int_0^1 k_1(x, y)\, g(y)\,dy. \tag{6.4}$$

We have the familiar series ([12] p. 228)

$$g(x) = -\lambda c(g) \sum \frac{a_j}{1 - \lambda/\lambda_j} f_j(x), \qquad \lambda \neq \lambda_j. \tag{6.5}$$

This is not a solution to (6.3), since g appears on both sides. But we may multiply both sides by $\varphi_1(x)$ and integrate; termwise integration of the series is easily justified. We obtain

$$c(g) = -c(g)\lambda \sum a_j^2/(1 - \lambda/\lambda_j).$$

Hence either g is such that $c(g)$ vanishes, or else λ must be a root of the function

$$P(\lambda) = 1 + \lambda \sum a_j^2/(1 - \lambda/\lambda_j).$$

If $\lambda \neq \lambda_j$, then $c(g)$ cannot be zero, since otherwise (6.4) would be a homogeneous equation with a nontrivial solution existing for λ not an eigenvalue of the kernel. Consequently, the only possible values of λ for which (6.3) has a solution are either the roots of $P(\lambda)$ or else $\lambda = \lambda_j$. That is, λ must be a root of $D_1(\lambda) = d_1(\lambda)P(\lambda)$.

Now $P(\lambda)$ is analytic save for possible simple poles at $\lambda = \lambda_j$. Hence $D_1(\lambda)$ is an entire function of λ with $D_1(0) = 1$, and is of genus zero. We note that

$$\sum 1/\lambda_j = \int_0^1 k_1(s, s)\,ds < \infty, \qquad \sum a_j^2 = \int_0^1 \varphi_1^2(t)\,dt < \infty.$$

Consequently, if for any root $\lambda = \lambda^*$ of $D_1(\lambda)$, (6.3) has a solution $g^*(x)$ with $\int_0^1 g^{*2}(x)\,dx = 1$, then $D_1(\lambda)$ is indeed the Fredholm determinant of the kernel $\rho_1(x, y) = k_1(x, y) - \varphi_1(x)\varphi_1(y)$ in (6.3).

Let now $\lambda = \lambda^*$ be a root of $D_1(\lambda) = d_1(\lambda)P(\lambda)$. We have the following two cases

(1) $\lambda^* \neq \lambda_j$; (2) $\lambda^* = \lambda_j$, $a_j = 0$.

In case (2), if $\lambda^* = \lambda_j$ and $a_j \neq 0$, then $D_1(\lambda^*) \neq 0$, since the roots of $d_1(\lambda)$ are simple.

In case (1), λ^* is a root of $P(\lambda)$; we show immediately that λ^* is real and simple. If we substitute the function

$$g^*(x) = -\sum \frac{a_j f_j(x)}{1 - \lambda^*/\lambda_j} \Big/ \sqrt{\sum \frac{a_j^2}{(1 - \lambda^*/\lambda_j)^2}} \tag{6.6}$$

in (6.3), we verify easily that it satisfies the equation. Since the kernel in (6.3) is symmetric, λ^* is real. Then, since

$$P'(\lambda) = \sum \left(\frac{a_j}{1 - \lambda/\lambda_j}\right)^2, \qquad \lambda \neq \lambda_j \text{ if } a_j \neq 0,$$

is positive for real λ, λ^* is simple; the case when $\int_0^1 \varphi_1^2(x)\,dx = 0$ is disposed of easily, of course. Thus, for any root under case (1) we have the solution $g^*(x)$ given by (6.6).

In case (2) there are two subcases, $P(\lambda^*) \neq 0$ and $P(\lambda^*) = 0$. If $P(\lambda^*) \neq 0$, then $\lambda^* = \lambda_j$ is a simple zero of $D_1(\lambda)$; in this case $f_j(x)$ is readily seen to satisfy (6.3), and $f_j(x)$ is orthogonal to $\varphi_1(x)$. If $P(\lambda^*) = 0$, then $D_1(\lambda)$ has a double root at $\lambda = \lambda^*$, and besides the solution $f_j(x)$ we have the solution $g^*(x)$ of (6.6), which is orthogonal to $f_j(x)$. Accordingly, to each root of $D_1(\lambda)$ we obtain solutions to (6.3) of the proper multiplicity. Theorem 6.2 is thus proved.

There is only a slight complication to prove an analogous theorem in the case when $d_1(\lambda)$ has repeated roots.

By plotting the function $P(\lambda)$ for real λ, it is easy to prove that $D_1(\lambda)$ has at most one negative zero λ_0^*; the condition for its existence is that $\sum_{j=1}^{\infty} \lambda_j a_j^2 > 1$, or that the series diverges. If the positive zeros are $0 < \lambda_1^* \leqq \lambda_2^* \leqq \cdots$, repeated according to their multiplicity, we verify easily that $\lambda_j^* \geqq \lambda_j$ for $j = 1, 2, \cdots$. This remark will be useful in the sequel.

In Theorem 6.2, in order to find $D_1(\lambda) = d_1(\lambda)P(\lambda)$, it is necessary to know the λ_j and the Fourier coefficients a_j of $\varphi_1(x)$. The following theorem gives an evaluation of $D_1(\lambda)$ through the solution of a nonhomogeneous integral equation without specifically bringing these quantities into evidence.

THEOREM 6.3 *Let $k_1(x, y)$ and $\varphi_1(x)$ be as in Theorem 5.2. Regarding s as a parameter only, let $u(t, s)$ be the solution of*

$$u(t, s) = \varphi_1(t)\varphi_1(s) + \lambda \int_0^1 k_1(\xi, t)u(\xi, s)\,d\xi. \tag{6.7}$$

Then the Fredholm determinant $D_1(\lambda)$ *for the kernel* $k_1(x, y) - \varphi_1(x)\varphi_1(y)$ *is*

$$D_1(\lambda) = d_1(\lambda)\left(1 + \lambda \int_0^1 u(t, t)\, dt\right).$$

The theorem follows quite simply from the series solution ([12], p. 228), using the previous notation,

$$u(t, s) = \sum_{j=1}^{\infty} \frac{a_j f_j(t)}{1 - \lambda/\lambda_j} \varphi_1(s)$$

so that $\int_0^1 u(t, t)\, dt = \sum a_j^2/(1 - \lambda/\lambda_j)$. The result follows by using Theorem 6.2.

For the analysis of the present problem, we have, from (6.1), $k(x, y) = \min(x, y) - xy$. In this case it is well known that

$$\lambda_j = \pi^2 j^2, \qquad f_j(x) = \sqrt{2} \sin \pi j x, \qquad d(\lambda) = (\sin \sqrt{\lambda})/\sqrt{\lambda}.$$

Using Theorem 6.2 we have, concerning the limiting ch.fn of C_n^2,

COROLLARY 6.4 *The function* $D(\lambda)$ *of Theorem* 6.1 *is*

$$D(\lambda) = \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}\left(1 + \lambda \sum_{j=1}^{\infty} \frac{a_j^2}{1 - \lambda/\pi^2 j^2}\right)$$

where $a_j = \sqrt{2} \int_0^1 \varphi(x) \sin \pi j x\, dx$.

It turns out to be possible to replace this expression by a quadrature, and we have

THEOREM 6.5 *The function* $D(\lambda)$ *of Theorem* 6.1 *is*

$$D(\lambda) = \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} - 2 \int_0^1 \int_0^t \varphi'(s)\varphi'(t) \cos \sqrt{\lambda}(1 - t) \cos \sqrt{\lambda}\, s\, ds\, dt. \tag{6.8}$$

To prove the theorem we first note that

$$\pi k a_k = \sqrt{2} \int_0^1 \varphi'(t) \cos k\pi t\, dt.$$

Hence, supposing initially that $\lambda \neq \pi^2 k^2$,

$$\begin{aligned} P(\lambda) &= 1 + \lambda \sum \frac{a_k^2}{1 - \lambda/\pi^2 k^2} \\ &= 1 + 2\lambda \int_0^1 \int_0^1 \varphi'(t)\varphi'(s) \sum_1^{\infty} \frac{\cos k\pi t \cos k\pi s}{\pi^2 k^2 - \lambda}\, dt\, ds \\ &= 1 + \int_0^1 \int_0^1 \varphi'(t)\varphi'(s)\left\{1 - \frac{\sqrt{\lambda}}{\sin \sqrt{\lambda}} L(t, s)\right\} dt\, ds \\ &= 1 - \frac{\sqrt{\lambda}}{\sin \sqrt{\lambda}} \int_0^1 \int_0^1 \varphi'(t)\varphi'(s) L(t, s)\, dt\, ds, \end{aligned}$$

$$L(t, s) = \begin{cases} \cos \sqrt{\lambda}(1 - t) \cos \sqrt{\lambda}\, s, & t \geqq s; \\ \cos \sqrt{\lambda}(1 - s) \cos \sqrt{\lambda}\, t, & t \leqq s. \end{cases}$$

Hence, since $D(\lambda) = (1/\sqrt{\lambda}) \sin \sqrt{\lambda}\ P(\lambda)$, we obtain the conclusion to the theorem, $L(t, s)$ being symmetric.

By using Theorem 6.3 we can obtain an alternative expression for $D(\lambda)$ in terms of the solution to a differential equation.

THEOREM 6.6 *Regarding λ and h as parameters, let $v = v(t, h)$ be the solution to*

$$v'' + \lambda v = h\varphi''(t), \qquad v(0, h) = v(1, h) = 0. \tag{6.9}$$

Then the function $D(\lambda)$ of Theorem 6.1 is

$$D(\lambda) = (1/\sqrt{\lambda}) \sin \sqrt{\lambda} \left(1 + \lambda \int_0^1 v(t, \varphi(t))\, dt\right). \tag{6.10}$$

In fact, the integral equation (6.7) is equivalent to the differential equation, with s parametric,

$$v'' + \lambda v = \varphi(s)\varphi''(t),$$

which vanishes at $t = 0$ and $t = 1$. Hence, using Theorem 6.3 we obtain (6.10) immediately.

The result of Theorem 6.5 follows directly from Theorem 6.6, for in solving (6.9) we determine the Green's function $G(\xi, t)$ for the differential expression $v'' + \lambda v$ over the interval $(0, 1)$. This is

$$G(\xi, t) = \begin{cases} \dfrac{\sin \sqrt{\lambda}\xi \sin \sqrt{\lambda}(1 - t)}{\sqrt{\lambda} \sin \sqrt{\lambda}}, & t > \xi; \\ \dfrac{\sin \sqrt{\lambda}t \sin \sqrt{\lambda}(1 - \xi)}{\sqrt{\lambda} \sin \sqrt{\lambda}}, & t < \xi. \end{cases}$$

Hence $v = h \int_0^1 G(\xi, t)\varphi''(\xi)\, d\xi$, so that (6.10) will give (6.8) after integration by parts.

The above results give various methods of obtaining $D(\lambda)$ and the ch.fn. of C^2, which is $[D(2it)]^{-1/2}$. The distribution can be obtained by the inversion formula

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} [D(2it)]^{-1/2}\, dt,$$

$$U(x) = 1 - \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \int_{\lambda_{2k-1}}^{\lambda_{2k}} \frac{e^{-xy/2}\, dy}{\sqrt{-y^2 D(y)}},$$

where[2] $U(x) = \int_0^x u(y)\, dy$ is the cdf of C^2, and $0 < \lambda_1 < \lambda_2 < \cdots$ are the (simple) zeros of $D(\lambda)$.

7. Properties of the C_n^2 test. A matter of central interest is whether the C_n^2 test has certain distribution-free properties. For example, it is important to know if it is parameter-free, that is, if the limiting distribution of C_n^2 does not

[2] The factor $(-1)^{k-1}$ is missing throughout Smirnov's [14] analysis.

depend on the parameter θ in $F(x; \theta)$. Clearly the test will be parameter-free if and only if $\varphi(u)$ given by (6.2) does not depend on θ; only in this case do we really have a usable test. This turns out to be the case when θ is a location or scale parameter.

THEOREM 7.1 *If $R(x)$ is a* cdf *and if $F(x; \theta) = R(x - \theta)$ or $F(x; \theta) = R(x/\theta)$, for $\theta > 0$, then $\varphi(u)$ is independent of θ and the C_n^2 test is parameter-free.*

To prove the theorem, we need only compute $\varphi(u)$ as given by (6.2). Let $r(x)$ be the density corresponding to $R(x)$. Then, when $F(x; \theta) = R(x - \theta)$, easy calculation shows

$$\varphi(u) = -\left(\int_{-\infty}^{\infty} [r'^2(x)/r(x)]\, dx\right)^{-1/2} r(R^{-1}(u)). \tag{7.1}$$

When $F(x; \theta) = R(x/\theta)$,

$$\varphi(u) = -\left(\int_{-\infty}^{\infty} x^2[r'^2(x)/r(x)]\, dx - 1\right)^{-1/2} R^{-1}(u) r(R^{-1}(u)), \tag{7.2}$$

where $R^{-1}(u)$ is the inverse of $R(x)$.

Another case of interest is $F(x; \theta) = [R(x)]^\theta$ for $\theta > 0$. Here easy computation shows

$$\varphi(u) = u \log u. \tag{7.3}$$

Likewise when $F(x; \theta) = 1 - [1 - R(x)]^\theta$ for $\theta > 0$, we get

$$\varphi(u) = -(1 - u) \log (1 - u).$$

In these latter two cases, $\varphi(u)$ does not even depend on R, so the test is distribution-free over this class of cdf's.

In general, however, $\varphi(u)$ does depend on θ. Consider, for example,

$$F(x; \theta) = \theta R^2(x) + (1 - \theta)R(x), \qquad 0 \leqq \theta \leqq 1.$$

For this, simple calculation gives

$$\varphi(u) = \left(\frac{\tanh^{-1}\theta}{\theta} - 1\right)^{-1/2} \left(u - \frac{-1 + \theta + \sqrt{(1 - \theta)^2 + 4u\theta}}{2\theta}\right).$$

Although not depending on R, this does depend on θ. At $\theta = 0$ we get $\varphi(u) = \sqrt{3}\, u(1 - u)$.

In Section 6 we have given various methods of obtaining the Fredholm determinant $D(\lambda)$ for the kernel $\rho(u, v) = \min(u, v) - uv - \varphi(u)\varphi(v)$. The eigenvalues, λ_k for $k = 1, 2, \cdots$, of this determinant are positive since ρ is a correlation function; that is, the case of one negative root λ_0 of $D(\lambda)$ discussed at the end of the proof of Theorem 6.2 does not occur. For the limiting semi-invariants of C_n^2 we then have [6]

$$K_j = 2^{j-1}(j - 1)! \sum 1/\lambda_n^j, \qquad j = 1, 2, \cdots, \tag{7.4}$$

or $K_j = 2^{j-1}(j-1)! \int_0^1 \rho_j(u, u)\, du$, where $\rho_j(u, v)$ is the jth iterate of the kernel $\rho(u, v)$. Thus the mean K_1 and variance K_2 of C^2 are, for example,

$$\mu = K_1 = \int_0^1 \rho(u, u)\, du = \int_0^1 [s(1-s) - \varphi^2(s)]\, ds = \tfrac{1}{6} - \int_0^1 \varphi^2(u)\, du, \tag{7.5}$$

$$\begin{aligned} d^2 = K_2 &= 2 \int_0^1 \int_0^1 [\min(u, v) - uv - \varphi(u)\varphi(v)]^2\, du\, dv \\ &= \tfrac{1}{45} - 8 \int_0^1 \varphi(t)(1-t) \int_0^t s\varphi(s)\, ds\, dt + 2(\mu - \tfrac{1}{6})^2. \end{aligned} \tag{7.6}$$

These expressions may be compared with the corresponding expressions for the case when there is no parameter to be estimated, that is, the W_n^2 test of the hypothesis (1.2). From the remark at the end of Theorem 6.2 we conclude that $\lambda_k \geqq \pi^2 k^2$, so that from (7.4) it follows that *all the semi-invariants (in particular the mean and variance) for C^2 are less than the corresponding semi-invariants for W^2.*

Expressing now W^2 and C^2 as a weighted sum of χ^2 variables, we have

$$W^2 = \sum G_j^2/\pi^2 j^2, \qquad C^2 = \sum G_j^2/\lambda_j,$$

where $G_1, G_2, \cdots$ are independent Gaussian random variables with mean 0 and variance 1. All the weights $1/\lambda_j$ in C^2 are less than the corresponding weights $1/\pi^2 j^2$ in W^2; this is the analogue to the reduction of the number of "degrees of freedom" in the usual χ^2 test. It is accounted for by the fact that there is additional freedom in fitting the empirical and theoretical cdf's because of the estimated parameter θ.

In particular, by (7.5) the mean of C^2 is less than the mean for W^2, which is $\frac{1}{6}$, by the factor $\int_0^1 \varphi^2(u)\, du$. This dimensionless quantity gives an idea of the "diameter" of the family $F(x; \theta)$ with respect to θ.

8. Illustrative examples.

A. Location parameter. Let $F(x; \theta)$ be of the form

$$F(x; \theta) = R(x - \theta), \qquad f(x; \theta) = r(x - \theta),$$

for a cdf $R(x)$ whose density is $r(x) = R'(x)$. Then denoting

$$\sigma = \left(\int_{-\infty}^{\infty} [r'^2(x)/r(x)]\, dx \right)^{-1/2}$$

we have by (7.1), $\varphi(u) = -\sigma r(R^{-1}(u))$. Since

$$\varphi(R(y)) = -\sigma r(y), \qquad \varphi'(R(y)) = -\sigma r'(y)/r(y),$$

we get by (7.5), (7.6), and (6.8)

$$\mu = \tfrac{1}{6} - \sigma^2 \int_{-\infty}^{\infty} r^3(x)\, dx,$$

$$d^2 = \tfrac{1}{45} - 8\sigma^2 \int_{-\infty}^{\infty} [1 - R(y)] r^2(y)\, dy \int_{-\infty}^{\infty} R(x) r^2(x)\, dx + 2(\mu - \tfrac{1}{6})^2$$

$$D(\lambda) = (\sin \sqrt{\lambda})/\sqrt{\lambda} - 2\sigma^2 \int_{-\infty}^{\infty} \cos [\sqrt{\lambda}(1 - R(y))] r'(y)\, dy \int_{-\infty}^{y} \cos [\sqrt{\lambda}\, R(x)] r'(x)\, dx.$$

Unfortunately, even in simple cases it does not generally appear possible to evaluate $D(\lambda)$ further, let alone invert the ch.fn. $[D(2it)]^{-1/2}$ of C^2. This seems to be true for the normal distribution, that is, $r(x) = (1/\sqrt{2\pi}) \exp(-\frac{1}{2}x^2)$.

As a special case, let $F(x; \theta)$ be the Cauchy distribution $F(x; \theta) = R(x - \theta)$. Then

$$R(x) = \tfrac{1}{2} + (1/\pi) \tan^{-1} x = u;$$

$$r(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \qquad x = -\cot \pi u = R^{-1}(u)$$

$$\sigma^2 = 2, \qquad \varphi(u) = (-\sqrt{2}/\pi) \sin^2 \pi u.$$

Consequently $\mu = \frac{1}{6} - 1/4\pi^2$, and

$$D(\lambda) = \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} + \left(\frac{4\pi}{4\pi^2 - \lambda}\right)^2 (1 - \cos \sqrt{\lambda}).$$

Another case in which $D(\lambda)$ can be evaluated explicitly is $R(x) = \frac{1}{2} + \frac{1}{2} \tanh x$, in which case $\varphi(u) = \sqrt{3}\, u(1 - u)$.

B. Scale parameter. Let $F(x; \theta) = R(x/\theta)$ for $\theta > 0$, and $r(x) = R'(x)$. Put

$$\sigma = \left(\int_{-\infty}^{\infty} x^2 [r'^2(x)/r(x)]\, dx \right)^{-1/2}$$

so that, by (7.2), $\varphi(u) = -\sigma R^{-1}(u)\, r(R^{-1}(u))$. Since

$$\varphi(R(y)) = -\sigma y r(y), \qquad \varphi'(R(y)) = -\sigma(y\, r'(y)/r(y) + 1),$$

we obtain by (7.5), (7.6), and (6.8)

$$\mu = \tfrac{1}{6} - \sigma^2 \int_{-\infty}^{\infty} y^2 r^3(y)\, dy,$$

$$d^2 = \tfrac{1}{45} - 8\sigma^2 \int_{-\infty}^{\infty} [1 - R(y)]\, y\, r^2(y) \int_{-\infty}^{y} R(x)\, x\, r^2(x)\, dx\, dy + 2(\mu - \tfrac{1}{6})^2,$$

$$D(\lambda) = (\sin \sqrt{\lambda})/\sqrt{\lambda} - 2\sigma^2 \int_{-\infty}^{\infty} \cos \sqrt{\lambda}\, [1 - R(y)]\, [y\, r'(y) + r(y)] \int_{-\infty}^{y} \cos \sqrt{\lambda} R(x) [x r'(x) + r(x)]\, dx\, dy.$$

Consider again the case of the Cauchy distribution, where

$$R(x) = \tfrac{1}{2} + \frac{1}{\pi}\tan^{-1} x = u, \qquad r(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad x = -\cot \pi u.$$

The above formulas give

$$\sigma = \sqrt{2}, \qquad \varphi(u) = (1/\sqrt{2\pi}) \sin 2\pi u,$$

$$\mu = \tfrac{1}{6} - \frac{1}{4\pi^2}, \qquad D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\frac{1}{1-\lambda/4\pi^2}.$$

In this case $D(\lambda)$ has zeros at $\lambda = \pi^2 j^2$, except for $j = 2$. In the representation $C^2 = \sum^* G_j^2/\pi^2 j^2$, where $G_1, G_2, \cdots$ are independent Gaussian random variables with mean 0 and variance 1, the star on the summation means to exclude the term $j = 2$. Thus C^2 and W^2 differ only in that W^2 has this term included. This is a genuine case of a loss of a "degree of freedom", that is, one less term in the sum of the squares.

C. Exponential parameter. Suppose, as before, that $F(x;\theta) = [R(x)]^\theta$ for $\theta > 0$. Then, as in (7.3), $\varphi(u) = u \log u$ and

$$\mu = \tfrac{1}{6} - \tfrac{2}{27} \cong .092593, \qquad d^2 = \tfrac{1}{45} - \tfrac{88}{1125} + \tfrac{16}{324} + \tfrac{8}{729} \cong .0043566,$$

$$D(\lambda) = (\sin\sqrt{\lambda})/\sqrt{\lambda} - 2\int_0^1 (1+\log t)\cos\sqrt{\lambda}\,(1-t) \cdot \int_0^t (1+\log s)\cos\sqrt{\lambda}\, s\, ds\, dt.$$

9. Further considerations.

A. Incorporation of a weight function. If, instead of the C_n^2 statistic defined by (1.4), we use a weight function $\psi(x) \geqq 0$ as in [6], and define a new test statistic C_n^{*2} by

$$C_n^{*2} = n\int_{-\infty}^{\infty} [F_n(x) - F(x;\hat{\theta}_n)]^2 \psi(F(x;\hat{\theta}_n))\, dF(x;\hat{\theta}_n),$$

we can carry through a theory paralleling that in the preceding pages. Here $\psi(x)$ must have a bounded first derivative in $0 \leqq x \leqq 1$, and we should be led to the evaluation of the distribution of $C^{*2} = \int Y^2(u)\psi(u)\, du$, where $Y(u)$ is the process defined in Theorem 4.1. In this case we will need to use the results for $D_1(\lambda)$ given in Theorems 6.2 and 6.3, using

$$\rho_1(s,t) = k(s,t)\sqrt{\psi(s)\psi(t)}, \qquad \varphi_1(s) = \varphi(s)\sqrt{\psi(s)},$$

where $k(s,t)$ is given by (3.8) and $\varphi(s)$ is as in (4.4). A method for evaluating the function $d_1(\lambda)$ in Theorems 6.2 and 6.3 is given in [6] for $\rho_1(s,t)$ of this form. Then the ch.fn. for C^{*2} is $[D_1(2it)]^{-1/2}$.

B. A Kolmogoroff statistic. In place of C_n^2 we can consider

$$K_n = \sup_{-\infty<x<\infty} \sqrt{n}\,|F_n(x) - F(x;\hat{\theta}_n)|,$$

following the test proposed by Kolmogoroff (cf. [6]). The limiting distribution of K_n is the same as the distribution of

$$K = \sup_{0<t<1} |Y(t)|,$$

where $Y(t)$ is the process of Theorem 4.1; that is, $Y(u)$ is Gaussian with mean 0 and covariance $\rho(s, t) = \min(s, t) - st - \varphi(s)\varphi(t)$. Unfortunately $Y(u)$ is not Markoffian and the usual methods of determining the distribution of K by diffusion theory will not work. However, in the representation of $Y(u)$ by (4.8), the processes $Y(u)$ and $Z(u)$ are jointly Markoffian and a two dimensional diffusion equation could be used.

Grenander and Rosenblatt [13] were led to an absorption probability for a Gaussian process with covariance $\min(u, v) + g(u)g(v)$ in the estimation of the spectrum of a stationary process. In their case the covariance is the sum of two positive definite kernels, rather than difference, as in $\rho(s, t)$, which simplifies matters considerably in calculating distributions.

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CHARACTERIZATIONS OF COMPLETE CLASSES OF TESTS OF SOME MULTIPARAMETRIC HYPOTHESES, WITH APPLICATIONS TO LIKELIHOOD RATIO TESTS[1]

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1. Summary. For the problem of testing a simple hypothesis on a density function of the form $f_\theta(e) = \exp\{\psi_0(\theta) + \sum_1^k \psi_i(\theta)t_i(e) + t_0(e)\}$, explicit characterizations are given of a minimal essentially complete class of tests, the minimal complete class, and the closure of the class of Bayes' solutions, under certain assumptions. Applications are made to discrete distributions of the above form and to some problems of testing composite hypotheses. The likelihood ratio tests of these hypotheses are characterized and shown to be admissible under certain assumptions.

2. Introduction. Consider a probability density function of the form

$$g_\psi(e) = \exp\left\{\psi_0 + \sum_1^k \psi_i t_i(e) + t_0(e)\right\},$$

with e a sample point in an n-dimensional Euclidean space, and $\psi = (\psi_1, \cdots, \psi_k)$ a parameter in a subset Ψ of a k-dimensional Euclidean space. It can be shown [1] that $t = (t_1, \cdots, t_k) = [t_1(e), \cdots, t_k(e)]$ is a sufficient statistic which admits a density

$$p_\psi(t) = \exp\left\{\psi_0 + \sum_1^k \psi_i t_i + t_0\right\}$$

with respect to a measure in a k-dimensional Euclidean space, and that the family of such densities with $\psi \varepsilon \Psi$ is strongly complete. A family M of measures μ_θ on T is called *strongly complete* [1] if $\int_T f(t)d\mu_\theta = 0$ a.e. in Lebesgue measure on Ψ implies $f(t) = 0$ a.e. M.

We shall assume that $p_\psi(t)$ is a density with respect to Lebesgue measure unless the contrary is specified. Let T denote the set of values of t for which $p_\psi(t) > 0$ for some fixed $\psi \varepsilon \Psi$; we assume here that T does not vary with ψ.

We shall consider the problem of testing a hypothesis $H_0 : \psi \varepsilon \omega^0$ against an alternative $H_1 : \psi \varepsilon \omega'$, where ω^0 and ω' are disjoint subsets of Ψ. For the one-parameter case ($k = 1$), complete class results have been obtained by Lehmann [2] and by Blackwell, Girshick, and Rubin (cf. [3]), and certain minimax results have been obtained by Allen [4]. The present paper contains generalizations of Lehmann's results. We shall be concerned in the following with the case in which ω^0 consists of a point ψ^0 and $\omega' = \Psi - \omega^0$, except where otherwise specified.

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For all statistical purposes we can, as shown in [5], restrict our consideration to tests of H_0 each characterized by a measurable decision function $\delta = \delta(t)$, where $0 \leqq \delta(t) \leqq 1$, defined on T, such that when t is observed the test rejects H_0 with probability $\delta(t)$.

For any test δ, let $\beta_\delta(\psi)$ denote the probability that the test will accept H_0 when ψ is the parameter point; that is $\beta_\delta(\psi) = 1 - \int_T p_\psi \delta(t)\, dt$. We define the risk function $r_\delta(\psi)$ of a test δ as follows:

$$r_\delta(\psi) = \begin{cases} \beta_\delta(\psi) & \psi \in \omega', \\ 1 - \beta_\delta(\psi) & \psi \in \omega^0, \\ 0 & \psi \in (\Psi - \omega^0 - \omega'). \end{cases}$$

Let $\xi(\psi)$ denote a cumulative distribution function defined on Ψ. A Bayes' solution with respect to $\xi(\psi)$ is a test δ_ξ such that the "Bayes' risk" $r_\delta(\xi) = \int_\Psi r_\delta(\psi)\, d\xi(\psi)$ is minimized by taking $\delta = \delta_\xi$. Let B denote the set of all Bayes' solutions; that is $\delta \in B$ if and only if there exists a ξ such that δ is a Bayes' solution with respect to ξ. Let V' denote the set of all tests δ such that δ is non-randomized (i.e., $\delta(t)$ is 0 or 1 only) and the set A on which $\delta(t) = 0$ (the test's acceptance region) is the common part of T and some open convex set in k-dimensional Euclidean space. Let V denote the set of all tests δ such that $\delta \in V$ if and only if there exists a $\delta' \in V'$ such that the set on which $\delta'(t) \neq \delta(t)$ has Lebesgue measure zero.

The following are the usual definitions of some terms which will be used below: A class of tests is called *complete* if for every test δ outside the class there is a test δ' in the class which is uniformly better, that is such that $r_{\delta'}(\psi) \leqq r_\delta(\psi)$ for all ψ and $r_{\delta'}(\psi) < r_\delta(\psi)$ for some ψ. A class of tests is called *minimal complete* if it is complete but has no proper subset which is complete. A class of tests is called *essentially complete* if for every test δ outside the class there is a test δ' in the class such that $r_{\delta'}(\psi) \leqq r_\delta(\psi)$ for all ψ. A test is called *admissible* if there exists no uniformly better test. A class of tests is called *minimal essentially complete* if it is essentially complete but has no proper subclass with this property. A test δ will be called *uniformly as good* as a test δ' if $r_\delta(\psi) \leqq r_{\delta'}(\psi)$ for all ψ.

3. Characterizations of complete and essentially complete classes. The following theorem has been proved by Reiersøl [7].

THEOREM 1. *B is a subset of V.*

PROOF. For any ξ, let γ be the saltus of ξ at $\psi = \psi^0$. Then

$$\begin{aligned} r_\delta(\xi) &= \int_\Psi r_\delta(\psi)\, d\xi(\psi) = \int_\Psi \beta_\delta(\psi)\, d\xi(\psi) + \gamma(1 - 2\beta_\delta(\psi^0)) \\ &= \int_\Psi \left[\int_T \{1 - \delta(t)\} p_\psi(t)\, dt \right] d\xi(\psi) + \gamma - 2\gamma \int_T \{1 - \delta(t)\} p_{\psi^0}(t)\, dt \\ &= \gamma + \int_T \left[\left(\int_\Psi p_\psi(t)\, d\xi(\psi) \right) - 2\gamma p_{\psi^0}(t) \right] \{1 - \delta(t)\}\, dt. \end{aligned}$$

Let $Q(t) = \int_\Psi p_\psi(t)\, d\xi(\psi) - 2\gamma p_{\psi_0}(t)$, so that

$$r_\delta(\xi) = \int_T Q(t)(1 - \delta(t))\, dt + \gamma.$$

Clearly $r_\delta(\xi)$ is minimized by $\delta = \delta_\xi$ such that

$$\delta_\xi(t) = \begin{cases} 0 & t \in A_\xi = \{t \mid Q(t) < 0\}, \\ 1 & \text{all other } t. \end{cases}$$

But $Q(t) < 0$ is equivalent to

$$\begin{aligned} 2\gamma &> \int_\Psi \left(\frac{p_\psi(t)}{p_{\psi_0}(t)}\right) d\xi(\psi) \\ &= \int_\Psi \exp\left\{\sum_{i=1}^{k} (\psi_i - \psi_i^0)t_i + (\psi_0 - \psi_0^0)\right\} d\xi(\psi) \\ &= R(t), \text{ say.} \end{aligned}$$

Thus $A_\xi = \{t \mid R(t) < 2\gamma, \quad t \in T\}$.

Now for $0 < \lambda < 1$ and for any real u and v, we have

$$e^{\lambda u + (1-\lambda)v} \leqq \lambda e^u + (1 - \lambda)e^v,$$

with strict inequality holding in case $u \neq v$. Thus if t' and t'' are distinct points in A_ξ, then for $0 < \lambda < 1$ we have

$$R(\lambda t' + [1 - \lambda]t'') < \lambda R(t') + (1 - \lambda)R(t'') < 2\gamma,$$

proving that $\delta_\xi \in V'$. Furthermore $R(t) = 2\gamma$ holds only on a set of Lebesgue measure zero. For

$$\frac{\partial^2 R(t)}{\partial t_j^2} = \int_\Psi (\psi_j - \psi_j^0)^2 \exp\left\{\sum_1^k (\psi_i - \psi_i^0)t_i + (\psi_0 - \psi_0^0)\right\} d\xi(\psi)$$

is nonnegative for all t, for each j. Moreover, if for any j and t, $\partial^2 R(t)/\partial t_j^2 = 0$, then $\Pr\{\psi_j - \psi_j^0 \mid \xi\} = 1$. If for some t we have $\partial^2 R(t)/\partial t_j^2 = 0$ for $j = 1, \cdots k$, then $\Pr\{\psi = \psi^0 \mid \xi\} = 1$.

In the latter case the Bayes' acceptance regions are T itself and all subsets which differ from T by sets of Lebesgue measure zero. In the remaining case we have that for each $t \in T$ there exists a j for which $\partial^2 R(t)/\partial t_j^2 > 0$. Hence $R(t) = 2\gamma$ holds at most on the boundary points of the convex acceptance region A_ξ, which constitute a set of measure zero.

Since $R(t) = 2\gamma$ holds only on a set of Lebesgue measure zero, every Bayes' solution with respect to ξ must differ from δ_ξ at most on a set of Lebesgue measure zero. Hence every Bayes' solution is in V. Q.E.D.

Wald ([6], Theorem 5.8) has proved that under assumptions which are satisfied by $p_\psi(t)$, an essentially complete class of tests is constituted by the closure

$\bar{B}$ of B in the sense of regular convergence, which is defined as follows: $\lim_{i\to\infty} \delta_i = \delta_0$ in the regular sense if

$$\lim_{i\to\infty} \int_R \delta_i(t)\, dt = \int_R \delta_0(t)\, dt$$

for any bounded subset R of the sample space T. If $\lim_{i\to\infty} \delta_i = \delta_0$ except on a set of Lebesgue measure zero, then it is clear that δ_i converges to δ_0 also in the regular sense. For the special case of $\delta_i \in V$ (but not in general) the following converse is also true.

LEMMA. *Let* $\lim_{i\to\infty} \delta_i = \delta_0$ *in the regular sense, where* $\delta_i(t) = 0$ *on a convex set* A_i, *and* $\delta_i(t) = 1$ *elsewhere. Then* $\lim_{i\to\infty} \delta_i = \delta_0$ *except on a set of Lebesgue measure zero. Furthermore* $\delta_0 \in V$, *and except on a set of Lebesgue measure zero,*

$$\delta_0(t) = \begin{cases} 0 & \text{on } a_0 = \lim_{i\to\infty} \bigcap_{j\geq i} A_j, \\ 1 & \text{elsewhere.} \end{cases}$$

PROOF. Let

$$A_0 = \lim_{i\to\infty} \bigcup_{j\geq i} A_j, \qquad a_0 = \lim_{i\to\infty} \bigcap_{j\geq i} A_j, \qquad D = A_0 - a_0.$$

It is clear that a_0 is convex. If $\lim_{i\to\infty} \delta_i(t)$ does not exist almost everywhere in Lebesgue measure, then $\mu(D) > 0$, where μ denotes Lebesgue measure. For infinitely many A_i we have $D \subset A_i$, and for infinitely many A_i we have $DA_i = 0$.

Let h be a point of D which is not a boundary point of a_0. Let H be the smallest convex set containing a_0 and h, and let k be an interior point of $H - a_0$. Such points h and k exist except in the trivial cases $\mu(D) = 0$ or $\mu(a_0) = 0$.

Let K be the convex cone consisting of the half-lines from k through the points of a_0. Let K^* be the "negative" cone consisting of the other halves of the same lines. Then $\mu(HK^*) > 0$. Since $h \in D$, we have $HK^* \subset A_i$ for infinitely many A_i, and $HK^*A_i = 0$ for infinitely many A_i. Let R be a bounded subset of HK^* with $\mu(R) > 0$. Then $\int_R \delta_i(t)\, dt = \mu(R) > 0$ for infinitely many i, and also $\int_R \delta_i(t)\, dt = 0$ for infinitely many i. Thus δ_i do not converge in the regular sense if $\mu(D) > 0$, proving the first assertion of the lemma.

If the δ_i converge in the regular sense, and hence $\mu(D) = 0$, the second assertion of the lemma follows from the fact that a_0 is convex.

From the lemma it follows that $\bar{V} = V$, where $\bar{V}$ denotes the closure of V in the regular sense. Since $B \subset V$ by Theorem 1, $\bar{B} \subset \bar{V} = V$. Since $\bar{B}$ is essentially complete we have

COROLLARY 1. *V is an essentially complete class.*

Since for every test in V there is a test in V' with identical risk function, we have

COROLLARY 2. *V' is an essentially complete class.*

The above-mentioned completeness property of the family of distributions $p_\psi(t)$ implies that two decision functions with identical risk functions differ at most on a subset of T with Lebesgue measure 0. Hence we have

COROLLARY 3. *V is a complete class.*

In the following sections it will be shown that, under certain additional assumptions, affirmative answers to the following questions hold.

1) Does $\bar{B}$ coincide with V?
2) Is V the minimal complete class?
3) Is V' a minimal essentially complete class?

In general V will not be minimal complete if ω' is a proper subset of $\Psi - \omega^0$. For example, consider the case in which $p_\psi(t) = (2\pi)^{-1/2} \exp\{-\frac{1}{2}(t_1 - \psi_1)^2\}$, with ω^0 and ω' consisting respectively of $\psi_1 = 0$ and $\psi_1 = 1$, and with $k = 1$. Here the minimal complete class is known to consist of the class of best tests of all sizes. The acceptance regions of these tests are all the intervals which are infinite to the left and all the sets which coincide with such intervals to within sets of Lebesgue measure zero. Since these acceptance regions are a proper subclass of V, V is not minimal complete.

However, V is known to be minimal complete in some one-parameter cases. If we take the preceding example with ω' altered to consist of $\psi_1 = 1$ and $\psi_1 = -1$, the minimal complete class consists of all "two-tail" tests of all sizes ([6], pp. 136–138). The acceptance regions of these tests are all the intervals and all the sets differing from intervals by sets of Lebesgue measure zero, that is just the class V. Here V', the class of all open intervals, is clearly minimal essentially complete.

For other density functions $p_\psi(t)$ with $k = 1$ (that is, one-parameter distributions) the same properties can be proved for V and V' provided ω' contains a point $\psi_1' < \psi_1^0$ and a point $\psi_1'' > \psi_1^0$ (cf. [2]). While the following two sections give the same conclusions for $k > 1$, they necessarily require stronger restrictions on ω'. Thus the one-parameter results referred to will not all be contained as special cases.

4. Full characterization of $\bar{B}$. Hereafter "a test A" will mean a (nonrandomized) test δ with $\delta = 0$ on A and $\delta = 1$ on $T - A$.

ASSUMPTION 1. The subset ω' contains hyperspheres of arbitrarily large radii. That is, for some increasing unbounded sequence $d_1, d_2, \cdots$, all formal solutions ψ of the equations $d_j^2 = \sum_1^k (\psi_i - \psi_i^0)^2$ for $j = 1, 2, \cdots$ are contained in ω'. This assumption is satisfied if Ψ is k-dimensional Euclidean space and $\Psi - \omega'$ is bounded.

THEOREM 2. *If Assumption 1 is satisfied, $V = \bar{B}$.*

PROOF. Since $V = \bar{V}$, and by Theorem 1, $B \subset V$, we have $\bar{B} \subset V$. It remains to show that $V \subset \bar{B}$. The method of proof will be to show that every $A \in V$ is an essentially unique limit (in the regular sense) of a sequence of Bayes' solutions $\{A_{\xi_i}\}$.

LEMMA. *For every bounded convex set C, the test $A = CT$ is an essentially unique limit, in the regular sense, of a sequence of Bayes' solutions.*

PROOF OF LEMMA: Let A be the set-theoretic product of T and any bounded convex set C in k-dimensional Euclidean space. For each point $\psi \varepsilon (\Psi - \omega^0)$, let $\eta(\psi)$ be the quantity such that

$$\sum_{i=1}^{k}\left[\frac{\psi_i - \psi_i^0}{d(\psi)}\right] t_i + \eta(\psi) = 0, \qquad d(\psi) = +\sqrt{\sum_{i=1}^{k}(\psi_i - \psi_i^0)^2},$$

is the equation of a supporting hyperplane of C, and such that the left member of this equation is negative for t any interior point of C. For any $d > 0$, let

$$\omega_d = \{\psi \mid d(\psi) = d\}, \qquad \rho(\psi) = \exp\{(\psi_0^0 - \psi_0) + \eta(\psi)\, d(\psi)\}.$$

Let $\sigma_d = \int_{\omega_d} d\omega_d$, where $d\omega_d$ denotes an infinitesimal element of $(k-1)$-dimensional Lebesgue measure on ω_d. Let

$$\frac{1}{\gamma_d} = 1 + \frac{1}{\sigma_d}\int_{\omega_d} \rho(\psi)\, d\omega_d .$$

Then we define the cumulative distribution functions $\xi_d = \xi_d(\psi)$ as follows: ξ_d has a saltus γ_d at $\psi = \psi^0$, and for any subset ν of Ψ-ω^0,

$$\int_{\nu} d\xi_d(\psi) = \frac{\gamma_d}{\sigma_d}\int_{\nu\omega_d} \rho(\psi)\, d\omega_d .$$

Consider the Bayes' solution with respect to ξ_d given by

$$A_d = \{t \mid R_d(t) < 2\gamma_d, \quad t \varepsilon T\}$$

where, as in the proof of Theorem 1,

$$R_d(t) = \int_{\Psi} \exp\left\{\sum_{i=1}^{k}(\psi_i - \psi_i^0)t_i + (\psi_0 - \psi_0^0)\right\} d\xi_d(\psi).$$

Here

$$R_d(t) = \gamma_d + \frac{\gamma_d}{\sigma_d}\int_{\omega_d} \exp\left\{\sum_{i=1}^{k}(\psi_i - \psi_i^0)t_i + \eta(\psi)\, d(\psi)\right\} d\omega_d .$$

Thus

$$A_d = \left\{t \,\middle|\, 1 > \frac{1}{\sigma_d}\int_{\omega_d} \exp\left(d\cdot\left[\sum_{i=1}^{k}\frac{(\psi_i - \psi_i^0)}{d}\, t_i + \eta(\psi)\right]\right) d\omega_d, t \varepsilon T\right\}.$$

Now for t any interior point of A and $d > 0$, the exponent in the integrand in the preceding expression is d multiplied by a negative quantity which is numerically never less than $\Delta(t)$, for $\psi \varepsilon \omega_d$, where $\Delta(t)$ is the Euclidean distance from t to the closest boundary point of A. Thus as d increases the integrand approaches 0 uniformly over ω_d, and the inequality is satisfied for all sufficiently large values of d. Hence each interior point t of A is an interior point of A_d for all d exceeding some d_t.

For every exterior point t not a boundary point of A, let ω_d^t denote the subset of ω_d such that for each $\psi \varepsilon \omega_d^t$, the equation

$$\sum_{i=1}^{k}\left(\frac{\psi_i - \psi_i^0}{d(\psi)}\right) t_i + \eta(\psi) = 0$$

determines a plane which separates t from A; that is the left member of the preceding equation is positive for all $\psi \varepsilon \omega_d^t$. Since $\eta(\psi)$ is continuous, $\int_{\omega_d^t} d\omega_d = m(\omega_d^t)$, say, is positive. We can choose $\epsilon > 0$ such that for each $d > 0$, the subset ω_d^* of ω_d^t such that

$$\sum_{i=1}^{k} \left(\frac{\psi_i - \psi_i^0}{d(\psi)} \right) t_i + \eta(\psi) \geqq \epsilon, \qquad \psi \varepsilon \omega_d^*,$$

satisfies $\int_{\omega_d^*} d\omega_d \geqq \frac{1}{2} m(\omega_d^t)$. Hence in the inequality which defines A_d above, the exponent in the integrand is not less than $d \cdot \epsilon$ on a set of Lebesgue measure not less than $\frac{1}{2} m(\omega_d^t)$. Hence as d increases, the right member of the inequality approaches ∞, and the exterior point t of A is also an exterior point of A_d for all d exceeding some d_t'. Hence we have that A differs from $\lim_{d \to \infty} A_d$ at most by a subset of the boundary of A, which has Lebesgue measure zero, proving the lemma.

PROOF OF THEOREM 2. Since every unbounded convex set is an essentially unique limit, in the regular sense, of a sequence of bounded convex sets, and hence is included in the closure of any class which contains all bounded convex sets, Theorem 2 is proved.

EXAMPLE 1. Let $p_\psi(t) = (1/2\pi) \exp [-\frac{1}{2}\{(t_1 - \psi_1)^2 + (t_2 - \psi_2)^2\}]$. With Ψ as the (ψ_1, ψ_2) plane and $\omega^0 = \{(\psi_1^0, \psi_2^0)\}$, the subset $\omega' = \Psi - \omega^0$ contains all solutions (ψ_1, ψ_2) of $d^2 = (\psi_1 - \psi_1^0)^2 + (\psi_2 - \psi_2^0)^2$ for $d > 0$. Hence the convex sets in the (t_1, t_2) plane are an essentially complete class for testing H_0 : $(\psi_1, \psi_2) = (\psi_1^0, \psi_2^0)$. A similar conclusion will clearly hold when $p_\psi(t)$ is any multivariate normal distribution with known covariance matrix and unknown mean ψ.

5. Characterization of a minimal essentially complete class.

CASE 1: *Bounded acceptance regions.*

ASSUMPTION 2. t_0 is a continuous function of t.

Let $\mathfrak{D}_b$ denote the class of tests A of H_0 whose acceptance regions are bounded. Let V_b' denote the class of tests in V' with bounded acceptance regions.

THEOREM 3. *If A' and A'' are distinct tests in V_b', then neither test is uniformly as good as the other, if Ψ satisfies Assumptions* 1 *and* 2.

COROLLARY 1. *If T is bounded, and if Assumptions* 1 *and* 2 *are satisfied, then V' is a minimal essentially complete class.*

PROOF OF COROLLARY 1. When T is bounded, V_b' coincides with V' which is, by Corollary 2 of Theorem 1, an essentially complete class. But an essentially complete class, no member of which is uniformly as good as any other, is minimal essentially complete.

PROOF OF THEOREM 3. Let A', A'' be distinct tests in V_b'. Assume that

$$r_{A'}(\psi^0) = r_{A''}(\psi^0).$$

Then each of the sets $A' - A''$ and $A'' - A'$ has positive Lebesgue measure. Let t^* be any interior point of $A' - A''$, and let T' be any supporting hyperplane of A'' which separates t^* form A''.

Let the equation of T' be written $\sum_1^k a_i t_i + a_0 = 0$, with $\sum_1^k a_i^2 = 1$. Let $\Delta(t) = \sum_1^k a_i t_i + a_0$; that is, $\Delta(t)$ is the directed distance from T' to any point t. Let

$$A^* = \left\{ t \,\middle|\, \sum_1^k (t_i - t_i^*)^2 \leqq \tfrac{1}{4} |\Delta(t^*)|^2, \quad t \in A' \right\}.$$

Then A^* has measure $L(A^*) > 0$. For t' any point in A^*, and t'' any point of $\overline{A''}$, the closure of A'', we have

$$|\Delta(t') - \Delta(t'')| \geqq \tfrac{1}{2}|\Delta(t^*)| > 0.$$

Let $\psi^c = c(a_1, \cdots a_k)$ for any real c, and let

$$g(t', t'', c) = \log [p_{\psi^c}(t')/p_{\psi^c}(t'')] - (t_0' - t_0'') = c\sum a_i(t_i' - t_i'')$$

$$= \pm c\,|\Delta(t') - \Delta(t'')|.$$

Now by Assumption 1, ω' contains points ψ^c with $c \to \infty$ and with $c \to -\infty$. Hence we can choose a sequence $\psi^{c_1}, \psi^{c_2}, \cdots$ such that $\lim_{i\to\infty} |c_i| = \infty$ and all c_i are of the same sign as $\sum_1^k a_i(t_i' - t_i'')$, for $t' \in A^*$ and $t'' \in A''$. By Assumption 3, $|t_0' - t_0''|$ is continuous and hence bounded say by v, for $t' \in A^*$ and $t'' \in \overline{A''}$. But

$$\log[p_{\psi^{c_i}}(t')/p_{\psi^{c_i}}(t'')] = \pm c_i\,|\,\Delta(t') - \Delta(t'')\,| + (t_0' - t_0'')$$

$$> \tfrac{1}{2}\,|c_i| \cdot |\Delta(t^*)| - v.$$

Hence $\lim_{i\to\infty} [p_{\psi^{c_i}}(t')/p_{\psi^{c_i}}(t'')] = \infty$ uniformly for $t' \in A^*$ and $t'' \in \overline{A''}$. That is, for any $M > 0$ there exists a c' such that $p_{\psi^{c'}}(t')/p_{\psi^{c'}}(t'') > M$ for all $t' \in A^*$ and $t'' \in \overline{A''}$. Let

$$\underline{p} = \inf_{t' \in A^*} p_{\psi^{c'}}(t'), \qquad \bar{p} = \sup_{t'' \in A''} p_{\psi^{c'}}(t'');$$

then $\underline{p}/\bar{p} \geqq M$. Now

$$r_{A'}(\psi^{c'}) \geqq \int_{A^*} p_{\psi^{c'}}(t)\,dt \geqq L(A^*)\underline{p},$$

$$r_{A''}(\psi^{c'}) = \int_{A''} p_{\psi^{c'}}(t)\,dt \leqq L(A'')\bar{p},$$

where $L(A'')$ is the measure of A''. Hence

$$\frac{r_{A'}(\psi^{c'})}{r_{A''}(\psi^{c'})} \geqq \frac{L(A^*)\cdot\underline{p}}{L(A'')\cdot\bar{p}} \geqq \frac{L(A^*)}{L(A'')} \cdot M.$$

But M may be chosen arbitrarily large, in particular so large that

$$M > L(A'')/L(A^*).$$

In this case we obtain $r_{A'}(\psi) > r_{A''}(\psi)$ for $\psi =$ some ψ^{c_i}.

Therefore A' is not uniformly as good as A''. Interchanging A' and A'' in the preceding discussion, we conclude also that A'' is not uniformly as good as A'.

It remains to consider tests A' and A'' in V_b' such that

$$r_{A'}(\psi^0) \neq r_{A''}(\psi^0).$$

Assume $r_{A'}(\psi^0) < r_{A''}(\psi^0)$. Then $A' - A''$ has positive measure. The proof given above for the case $r_{A'}(\psi^0) = r_{A''}(\psi^0)$ then shows that there exists a $\psi \varepsilon \Psi$ such that $r_{A'}(\psi) > r_{A''}(\psi)$. Similarly if $r_{A'}(\psi^0) > r_{A''}(\psi^0)$, we can show that there exists a $\psi \varepsilon \Psi$ such that $r_{A'}(\psi) < r_{A''}(\psi)$. Q.E.D.

The method of proof of Theorem 3 can be applied directly to prove that no test not in V is uniformly as good as *any* test in V_b'. Hence we have

COROLLARY 2. *If Assumptions* 1 *and* 2 *are satisfied and* T *is bounded, then* V *is the minimal complete class.*

These results are used in connection with Example 1 in section 10, below.

6. Characterization of a minimal essentially complete class.

CASE 2. *The General Case.* ASSUMPTION 3. Let T' be the common part of T and any half-space, that is, $T' = \{t \mid \sum a_i t_i + a_0 \leqq 0,\ t \varepsilon T\}$, for arbitrary a_i's subject to $\sum a_i^2 = 1$. Let t' be any point in $T - T'$. Then there exists a sequence of points ψ^i in ω' such that, uniformly in some neighborhood of t',

$$\lim_{i \to \infty} \left[p_{\psi^i}(t') \Big/ \int_{T'} p_{\psi^i}(t)\, dt \right] = \infty.$$

THEOREM 4. *If Assumptions* 2 *and* 3 *are satisfied,* V' *is a minimal essentially complete class.*

PROOF. By Corollary 2 of Theorem 1, V' is essentially complete. It remains to show that, of any pair of distinct tests A' and A'' in V', neither test is uniformly as good as the other.

Given any such pair of tests, at least one of the sets $A' - A''$ and $A'' - A'$ has positive measure. Assume $A' - A''$ has positive measure, and let t^* be any interior point of $A' - A''$. Let $\Delta(t) = \sum a_i t_i + a_0$. Let

$$A^* = \left\{ t \,\middle|\, \sum_{i=1}^{k} (t_i - t_i^*)^2 \leqq |\Delta(t^*)|^2 \epsilon^2,\quad t \varepsilon A' \right\}.$$

Then for some ϵ, with $0 < \epsilon < 1$, we have by Assumption 3 that for arbitrarily large M, there exists i_M such that for all $t' \varepsilon A_\epsilon^*$,

$$p_{\psi^i}(t') \Big/ \int_{T'} p_{\psi^i}(t)\, dt > M, \qquad i \geqq i_M.$$

Let $p = \inf_{t \varepsilon A_\epsilon^*} p_{\psi^{i_M}}(t)$. Let $L(A_\epsilon^*)$ be the measure of A_ϵ^*. Then

$$p \geqq M \cdot \int_{T'} p_{\psi^{i_M}}(t)\, dt.$$

Now

$$r_{A'}(\psi^{i_M}) \geqq \int_{A_\epsilon^*} p_\psi i_M(t)\, dt \geqq L(A_\epsilon^*) \cdot p,$$

$$r_{A''}(\psi^{i_M}) \leqq \int_{T'} p_\psi i_M(t)\, dt.$$

Hence $r_{A'}(\psi^{i_M}) \geqq M \cdot L(A_\epsilon^*) \cdot r_{A''}(\psi^{i_M})$. But M may be chosen arbitrarily large, in particular so large that $M > 1/L(A_\epsilon^*)$, in which case we obtain

$$r_{A'}(\psi^{i_M}) > r_{A''}(\psi^{i_M}).$$

Thus A' is not uniformly as good as A''.

If $r_{A'}(\psi^0) = r_{A''}(\psi^0)$, we have that $A'' - A'$, as well as $A' - A''$, has positive measure. By interchanging A' and A'' in the preceding discussion we conclude also that A'' is not uniformly as good as A'. If $r_{A'}(\psi^0) < r_{A''}(\psi^0)$, then $A' - A''$ has positive measure and as above we conclude that $r_{A'}(\psi) > r_{A''}(\psi)$ for some ψ. Similarly if $r_{A'}(\psi^0) > r_{A''}(\psi^0)$, we conclude as above that $r_{A'}(\psi) < r_{A''}(\psi)$ for some ψ. Q.E.D.

The method of proof of Theorem 4 can be applied directly to prove that no test not in V is uniformly as good as *any* test in V'. Hence we have

COROLLARY. *If Assumptions* 2 *and* 3 *are satisfied, then* V *is the minimal complete class.*

EXAMPLE. Let $p_\psi(t) = (1/2\pi) \exp\{-\sum_1^2 \tfrac{1}{2}(x_i - \mu_i)^2\}$, and let H_0 specify $(\mu_1, \mu_2) = (0, 0)$. For any a_1, a_2, and a_0 such that $a_1^2 + a_2^2 = 1$, let

$$T' = \{(x_1, x_2) \mid a_1x_1 + a_2x_2 + a_0 \leqq 0\},$$

$$y = y(t) = (y_1, y_2) = (a_1x_1 + a_2x_2, -a_2x_1 + a_1x_2).$$

Then we may write $p_\psi(t) = (1/2\pi) \exp\{-\tfrac{1}{2}\sum_1^2 (y_i - \nu_i)^2\}$, with $H_0 : (\nu_1, \nu_2) = (0, 0)$, and $T' = \{(y_1, y_2) \mid y_1 \leqq -a_0\}$. Let $t' = (y_1, y_2)$ be any point not in T'; that is, $y_1' > -a_0$. Let

$$v = \frac{p_\psi(t')}{\int_{T'} p_\psi(t)\, dt}$$

$$= \frac{(1/2\pi) \exp\{-\tfrac{1}{2}[(y_1' - \nu_1)^2 + (y_2' - \nu_2)^2]\}}{(1/2\pi) \int_{-\infty}^{\infty} \left[\int_{-\infty}^{-a_0} \exp\{-\tfrac{1}{2}(y_1 - \nu_1)^2\}\, dy_1 \right] \exp\{-\tfrac{1}{2}(y_2 - \nu_2)^2\}\, dy_2}.$$

Proceeding as in [8], we set

$$\nu_2 = 0, \qquad u = -a_0 - \nu_1, \qquad \Delta = y_1' + a_0, \qquad h = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}y_2'^2).$$

Then $v = h \exp[-\tfrac{1}{2}(u + \Delta)^2] \Big/ \int_{-\infty}^{u} \exp[-\tfrac{1}{2}r^2]\, dr.$

Now for $\nu_1 > y_1'$,

$$u \equiv -a_0 - \nu_1 < u + \Delta < 0, \qquad u^2 > (u + \Delta)^2,$$

so that $\exp[-\tfrac{1}{2}(u + \Delta)^2] > \exp[-\tfrac{1}{2}u^2]$.

Similarly for $\nu_2 = 0$ and $\nu_1 > y_1'$,

$$v > h \cdot (\nu_1 + a_0), \qquad \lim_{\nu_1 \to \infty} v = \infty.$$

Thus Assumption 3 is satisfied, and by Theorem 4 we conclude that the open convex sets in the (x_1, x_2) plane constitute a minimal essentially complete class of acceptance regions (tests) of H_0.

7. Other cases. While Theorem 1 shows that $\bar{B} \subset V$ holds without restrictions on ω', an example above showed that even with $k = 1$ we do not have $\bar{B} = V$ unless ω' is sufficiently inclusive. For $k > 1$, when ω' does not satisfy Assumption 1 it is possible in some cases to use the methods of the preceding sections to obtain characterizations of minimal complete classes or at least characterizations of smaller complete classes than V.

Such possibilities will be illustrated in a special case in terms of Example 1 above. Let ω' now be

$$\{(\psi_1, \psi_2) \mid \psi_1 \geqq 0, \quad \psi_2 \geqq 0, \quad \psi_1 + \psi_2 > 0\};$$

that is ω' is the closed first quadrant with $\psi^0 = (0, 0)$ deleted. Then the method of Theorem 1 can be extended to show that every Bayes' acceptance region A_ξ has the property that if $(t_1, t_2) \varepsilon A_\xi$ and $t_1' \leqq t_1$, and $t_2' \leqq t_2$, then $(t_1', t_2') \varepsilon A_\xi$, with exceptions on sets of measure zero. The method of Theorem 3 can be adapted to prove that tests having the preceding property constitute the minimal complete class.

For cases in which ω' is bounded, in general no simple characterization of the minimal complete classes can be given. By extending the method of Theorem 1 it can be shown that when ω' is bounded the Bayes' acceptance regions A_ξ cannot approximate arbitrarily closely any bounded convex hyperpolygon. Thus a minimal essentially complete class consists of a proper subset of V' which can be described roughly as excluding all hyperpolygons with finite vertices and edges, and all acceptance regions which closely approximate such hyperpolygons; as the bounds on ω' are narrowed, more elements of V' are excluded.

For some common distributions of the form $p_\psi(t)$, Assumption 1 fails. Examples are $p_\psi(t)$ as determined by either

(a) $$g_\psi(e) = \frac{1}{(2\pi)^{N/2}\psi_2^N} \exp\left\{\frac{-1}{2\psi_2^2} \sum_{i=1}^{N} (x_i - \psi_1)^2\right\},$$

(b) $$g_\psi(e) = \frac{1}{(2\pi\psi_1\psi_2)^N} \exp\left\{\frac{-1}{2} \sum_{j=1}^{N} \left(\frac{x_{1j}^2}{\psi_1^2} + \frac{x_{2j}^2}{\psi_2^2}\right)\right\}.$$

For the problem of testing a simple hypothesis in (a), the density of a sample from a normal population with unknown mean and variance, Stein (in a private communication) has constructed a test which is in V, but which can be strictly improved on, showing that for this problem V is not a minimal complete class. For this testing problem, Walsh [9] has investigated the operating characteristics of three tests of H_0 as a basis for selecting tests for quality control and other applications. Only one of the three tests considered, that one whose acceptance

region is a rectangle in the $(\sum x_i^2, \sum x_i)$-plane, is contained in V and hence may be admissible; the remaining two tests are inadmissible.

For the class of problems considered in this paper, no general method is yet known for constructing a test which is uniformly better than a given inadmissible test. On the other hand, when some possible tests for a given application are to be examined on the basis of their operating characteristics, it seems advisable to restrict consideration to the smallest known complete class of tests, except possibly where computational difficulties make this impractical.

8. Tests of composite hypotheses. The methods of proof of the theorems and corollaries above can be used to reach certain conclusions concerning tests of composite hypotheses on distributions of the form of $p_\psi(t)$. Let H_0 now specify $\psi \in \omega^0$, where ω^0 consists of two or more points.

From the proof of Theorem 1 we conclude that V contains at least all of the Bayes' acceptance regions for H_0 corresponding to ξ's such that $\int_{\omega^0} d\xi(\psi)$ is equal to the saltus of ξ at some point ψ in ω^0; that is, $V \cdot \bar{B} \neq 0$.

From the proof of Theorem 2 we conclude that, when Assumption 1 is satisfied with ψ^0 some point in ω^0, then $V \subset \bar{B}$.

For any δ, $r_\delta(\psi)$ is a continuous function of ψ, except at ψ^0. Suppose that δ_0 is admissible for testing $\{\psi^0\}$ against $\omega' = \Psi - \{\psi^0\}$. Let ω^* be a subset of ω' having k-dimensional Lebesgue measure zero. Then δ_0 remains admissible for testing $\{\psi^0\}$ against $\Psi - \{\psi^0\} - \omega^*$, and also for testing $\{\psi^0\} + \omega^*$ against $\psi - \{\psi^0\} - \omega^*$.

Theorem 3 is valid when H_0 is composite. The proof of Theorem 4 shows that when Assumptions 2 and 3 are satisfied, neither of any pair of distinct tests in V' is uniformly as good as the other; hence when H_0 is composite, any essentially complete class will contain V' or an equivalent class.

Lehmann [10] has defined a class of "monotone critical regions." As some examples in the following section will show, when Assumptions 2 and 3 are satisfied these tests are admissible and in some cases are also likelihood ratio tests.

9. Likelihood ratio tests. A likelihood ratio test of a (simple or composite) hypothesis $H_0 : \psi \in \omega^0$ against an alternative $H_1 : \psi \in \omega'$ has the following form: Reject H_0 if and only if $\lambda(t) < K$, where

$$\lambda(t) = \sup_{\psi \in \omega^0} p_\psi(t) \Big/ \sup_{\psi \in (\omega^0 + \omega')} p_\psi(t)$$

and K is a constant, $0 \leqq K \leqq 1$. Let $A_K(\omega^0, \omega^0 + \omega')$ denote the acceptance region of such a test. Then

$$\begin{aligned} A_K(\omega^0, \omega^0 + \omega') &= \{t \mid \sup_{\psi \in \omega^0} p_\psi(t) \geqq K \cdot \sup_{\psi \in (\omega^0 + \omega')} p_\psi(t), \quad t \in T\} \\ &= \bigcup_{\psi'' \in \omega^0} \{t \mid p_{\psi''}(t) \geqq K \cdot \sup_{\psi \in (\omega^0 + \omega')} p_\psi(t), \quad t \in T\} \\ &= \bigcup_{\psi'' \in \omega^0} \bigcap_{\psi' \in (\omega^0 + \omega')} \{t \mid p_{\psi''}(t) \geqq K \cdot p_{\psi'}(t), \quad t \in T\}. \end{aligned}$$

[These relations are independent of the form of $p_\psi(t)$. They generalize a remark of Neyman and Pearson ([11], p. 282). They provide a convenient construction of likelihood ratio tests in some cases in which the usual calculus methods fail or are cumbersome, as will be illustrated in an example below.]

Assume now that $p_\psi(t)$ has the form assumed above. If ω^0 consists of the point ψ^0 then

$$A_K(\omega^0, \omega^0 + \omega') = \bigcap_{\psi' \varepsilon (\omega^0 + \omega')} \{t \mid p_{\psi^0}(t) \geqq K \cdot p_{\psi'}(t), \quad t \varepsilon T\}$$

$$= \bigcap_{\psi' \varepsilon (\omega^0 + \omega')} \{t \mid \sum_{i=1}^{k} (\psi_i^0 - \psi_i')t_i \geqq \log K - (\psi_0^0 - \psi_0'), \quad t \varepsilon T\}.$$

Thus $A_K(\omega^0, \omega^0 + \omega')$ is a set-theoretic product of T and a set of "linear half-spaces" of T. Hence $A_K \varepsilon V$. Hence we conclude that the likelihood ratio tests of simple hypotheses are admissible if Assumption 1 or Assumption 2 holds.

For certain cases in which H_0 is composite it will again turn out that

$$A_K(\omega^0, \omega^0 + \omega') \varepsilon V,$$

so that the likelihood ratio test is admissible under the appropriate assumptions.

EXAMPLE 1. Let $p_\psi(t) = (1/2\pi) \exp [-\frac{1}{2} \sum_1^2 (t_i - \psi_i)^2]$. If H_0 and H_1 are simple and $(\psi_1^0, \psi_2^0) = (0, 0)$, then

$$A_K(\omega^0, \omega^0 + \omega') = \left\{(t_1, t_2) \,\middle|\, \sum_{i=1}^{2} (-\psi_i' t_i) \geqq \log K - \tfrac{1}{2}(\psi_1'^2 + \psi_2'^2)\right\}.$$

This is a half-plane bounded by a line whose directed distance from the origin is $d = [\log K - \frac{1}{2}(\psi_1'^2 + \psi_2'^2)]/\sqrt{\psi_1'^2 + \psi_2'^2}$. If now we take

$$\omega' = \{(\psi_1, \psi_2) \mid \psi_1^2 + \psi_2^2 = \psi_1'^2 + \psi_2'^2\},$$

according to the relations above $A_K(\omega^0, \omega^0 + \omega')$ will be the common part of all half-planes whose boundaries have directed distance d from the origin; that is

$$A_K(\omega^0, \omega^0 + \omega') = \{(t_1, t_2) \mid t_1^2 + t_2^2 \leqq d\}.$$

If next we take ω' to be $\Psi - \omega^0$, where Ψ is the (ψ_1, ψ_2)-plane, $A_K(\omega^0, \omega^0 + \omega')$ will be the common part of a family of sets of the form $\{(t_1, t_2) \mid t_1^2 + t_2^2 \leqq d\}$ and so will itself be a set of this form. If finally we take ω^0 as any set in Ψ, with ω' its complement, we obtain that in this general case $A_K(\omega^0, \omega^0 + \omega') = A_K(\omega^0, \Psi)$, the set of points (t_1, t_2) whose distance from the set ω^0 does not exceed a certain constant c.

For example, if $\omega^0 = \{(\psi_1, \psi_2) \mid \text{one or both } \psi_i \leqq 0\}$, then $A_K(\omega^0, \Psi) = \{(t_1, t_2) \mid \text{one or both } t_i \leqq c\}$.

If $\omega^0 = \{(\psi_1, \psi_2) \mid \text{both } \psi_i \leqq 0\}$, then $A_K(\omega^0, \Psi) = \{(t_1, t_2) \mid (t_1 \leqq c, \quad t_2 \leqq 0)$, or $(t_1 \leqq 0, \quad t_2 \leqq c)$, or $(t_1 > 0, \quad t_2 > 0, \quad \sqrt{t_1^2 + t_2^2} \leqq c)\}$.

A sufficient (but not necessary) condition that $A_K(\omega^0, \Psi)$ be convex is that ω^0 be convex. If ω^0 is any convex set the methods of sections 7 and 8 show that the likelihood ratio test of H_0 is admissible.

EXAMPLE 2. Consider the problem of testing whether a given ranking of the means of k normal populations with known variances is correct. Let

$$p_\psi(t) = \frac{1}{(2\pi)^{k/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{k}(t_i - \psi_i)^2\right].$$

Let H_0 specify that $\psi_1 \geqq \psi_2 \geqq \cdots \geqq \psi_k$. For simplicity, let

$$k = 3, \qquad y_1 = (t_1 - t_2)/\sqrt{2}, \quad y_2 = (t_2 - t_3)/\sqrt{2}.$$

Then, as $y = (y_1, y_2)$ is a sufficient statistic for $\nu = (\psi_1 - \psi_2, \ \psi_2 - \psi_3)$, we may take

$$p_\nu(y) = (1/2\pi) \exp\left[-\tfrac{1}{2}[(y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2]\right],$$

and write $H_0 : \nu_i \geqq 0$ for $i = 1, 2$. Using the expressions above, it is easily seen that $A_K(\omega^0, \Psi)$ is the set of points (y_1, y_2) swept out by an ellipse of the form

$$(y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2 = c,$$

as its center (ν_1, ν_2) sweeps throughout the first quadrant; that is

$$\begin{aligned} A_K(\omega^0, \Psi) = \{(y_1, y_2) \mid &\text{for some } \nu_1 \geqq 0, \quad \nu_2 \geqq 0, \\ &(y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2 \leqq c\} \\ = \{(y_1, y_2) \mid &(y_1 \geqq -\sqrt{c}, \quad y_2 \geqq 0), \text{ or } (y_2 \geqq -\sqrt{c}, \quad y_1 \geqq 0), \\ &\text{or } (y_1 < 0, \quad y_2 < 0, \quad y_1^2 - y_1 y_2 + y_2^2 \leqq c)\}. \end{aligned}$$

As in the preceding example we see that the likelihood ratio tests of H_0 are admissible. Since this test's maximum Type I error $\tilde{\alpha}$ is attained when $\nu_1 = \nu_2 = 0$, by use of tables of the bivariate normal distribution the constant c can be selected to give $\tilde{\alpha}$ any desired value.

10. The discrete case. The developments of the preceding sections can also be carried through in the main when $p_\psi(t)$ is a discrete probability distribution function. Regular convergence in the general case is defined (as in [6], p. 134) as follows: $\lim_{i\to\infty} \delta_i = \delta_0$ in the regular sense if, for every bounded subset R of T,

$$\lim_{i\to\infty} \int_R \delta_i(t)\, d\mu(t) = \int_R \delta_0(t)\, d\mu(t).$$

The proof of Theorem 1 above, with $p_\psi(t)$ taken to be discrete, gives

THEOREM 1*. *Every essentially unique (in ξ measure) Bayes' acceptance region is contained in V'.*

We define the class of tests V^* as follows: $\delta \in V^*$ if and only if there exists an open convex set S such that

$$\delta(t) = \begin{cases} 0 & t \text{ an interior point of } S, \\ 1 & t \text{ an exterior point of } S. \end{cases}$$

Then the proof of Theorem 1 above together with Theorem 5.5 of [6] gives

COROLLARY 1*. *If ω' is closed and bounded, V^* is a complete class.*

Since V^* is closed in the sense of regular convergence, from Theorem 5.7 of [6] we obtain

COROLLARY 2*. *V^* is an essentially complete class.*

From the methods of proof of Theorem 2 above we obtain in the discrete case

THEOREM 2*. *V' is contained in $\bar{B}$ if Assumption 1 is satisfied.*

In the discrete case Assumption 2 is always satisfied trivially unless T contains at least one convergent sequence and its limit point. From the methods of proof of Theorems 3 and 4 above, we obtain

THEOREM 3*. *If Assumptions 1 and 2 are satisfied, and if A' and A'' are distinct tests in V_b', then neither test is uniformly as good as the other.*

A modification of Assumption 3 is necessary for the discrete case

ASSUMPTION 3*. For any T' and t' as defined in Assumption 3, ω' contains a sequence of points ψ^i such that

$$\lim_{i\to\infty} p_{\psi^i}(t')/\sum_{t\in T'} p_{\psi^i}(t) = \infty.$$

THEOREM 4*. *If Assumptions 2 and 3* are satisfied, then neither of any pair of distinct tests in V' is uniformly as good as the other.*

Finally, using the methods of proof of Theorems 3 and 4, and defining V_b^* as the subclass of V^* consisting of those tests δ for which the set $\{t \mid \delta(t) < 1\}$ is bounded, we obtain

THEOREM 3a*. *If Assumptions 1 and 2 are satisfied, then neither of any pair of distinct tests in V_b^*, at least one of which is in V_b', is uniformly as good as the other.*

COROLLARY. *If T is bounded and Assumptions 1 and 2 are satisfied, every test in V' is admissible.*

THEOREM 4a*. *If Assumptions 2 and 3* are satisfied, then neither of any pair of distinct tests in V^*, at least one of which is in V', is uniformly as good as the other.*

COROLLARY. *If Assumptions 2 and 3* are satisfied, then every test in V' is admissible.*

EXAMPLE 1. Let $\theta = (p_1, p_2)$, and

$$f_\theta(t) = \frac{n!\, p_1^{t_1} p_2^{t_2}(1 - p_1 - p_2)^{n-t_1-t_2}}{t_1! t_2! (n - t_1 - t_2)!}, \qquad \begin{matrix} t_1 = 0, 1, \cdots (n - t_2), \\ t_2 = 0, 1, \cdots n. \end{matrix}$$

We may write

$$f_\theta(t) = p_\psi(t) = \frac{n! \exp\{\psi_1 t_1 + \psi_2 t_2 - n \log (e^{\psi_1} + e^{\psi_2})\}}{t_1!\, t_2!\, (n - t_1 - t_2)!}$$

where $\psi_i = \log [p_i/(1 - p_1 - p_2)]$ for $i = 1, 2$. For

$$H_0: \quad (p_1, p_2) = (p_1^0, p_2^0), \quad p_1^0 > 0, \quad p_2^0 > 0, \quad p_1^0 + p_2^0 < 1,$$

T is bounded, and t_0 is, trivially, continuous on

$$T = \{(t_1, t_2) \mid t_1 = 0, 1, \cdots, (n - t_2); \quad t_2 = 0, 1, \cdots, n\}.$$

Hence Assumptions 1 and 2 are satisfied, and from the Corollary to Theorem 3a* we conclude that an admissible acceptance region for H_0 is given by the common part of T and any convex set.

EXAMPLE 2. Let x_1 and x_2 have independent Poisson distributions, with unknown respective means λ_1 and λ_2. Let $t = (x_1, x_2)$, and consider

$$H_0: \quad \theta = (\lambda_1, \lambda_2) = (\lambda_1^0, \lambda_2^0), \qquad \lambda_i > 0, \quad i = 1, 2.$$

Let $T = (x_1, x_2)$ with $x_i = 0, 1, \cdots$, for $i = 1, 2$. Let A be the common part of T and any convex set. Then as in the preceding example we conclude that A is admissible.

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ASYMPTOTIC SOLUTIONS OF THE COMPOUND DECISION PROBLEM FOR TWO COMPLETELY SPECIFIED DISTRIBUTIONS[1]

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1. Summary. A compound decision problem consists of the simultaneous consideration of n decision problems having identical formal structure. Decision functions are allowed to depend on the data from all n components. The risk is taken to be the average of the resulting risks in the component problems. A heuristic argument for the existence of good asymptotic solutions was given by Robbins ([1] Sec. 6) and was preceded by an example (component decisions between $N(-1,1)$ and $N(1,1)$) exhibiting, for sufficiently large n, a decision function whose risk was uniformly close to the envelope risk function of "simple" decision functions.

The present paper considers the class of problems where the components involve decision between any two completely specified distributions, with the risk taken to be the weighted probability of wrong decision. For all sufficiently large n, decision functions are found whose risks are uniformly close to the envelope risk function of "invariant" decision functions.

2. Statement and reduction of the problem. The problem of testing a simple statistical hypothesis against a simple alternative can be formulated as follows. Let x be a random variable (of arbitrary dimensionality) which is known to have one of the two distinct distribution functions $F(x, \theta)$ for $\theta = 0$ or 1. On the basis of a single observation on x (we consider only the nonsequential case) it is require to decide whether the true value of the unknown parameter θ is 0 or 1.

Statistical decision problems of the same formal structure often occur, or can be considered, in large groups. We shall, therefore, take as a single entity the following compound decision problem. Let n be a fixed positive integer and let $x_1, \cdots, x_n$ be independent random variables, each of which has the distribution function $F(x, \theta)$ with respective parameter values $\theta_1, \cdots, \theta_n$, with $\theta_i = 0$ or 1. Let $\mathbf{x} = (x_1, \cdots, x_n)$ denote the vector of observations and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$ the unknown vector of parameters; $\boldsymbol{\theta}$ is known to belong to the set Ω consisting of all 2^n possible vectors of n components, each 0 or 1. On the basis of $\mathbf{x}$ it is required to decide the true value of $\boldsymbol{\theta}$, which amounts to deciding for every $i = 1, \cdots, n$ whether $\theta_i = 0$ or 1.

Any vector of n functions $\mathbf{t} = (t_1(\mathbf{x}), \cdots, t_n(\mathbf{x}))$ is a (randomized) decision function for the compound decision problem if for $i = 1, \cdots, n$, $0 \leqq t_i(\mathbf{x}) \leqq 1$, and if the conditional probabilities, given $\mathbf{x}$, of deciding that $\theta_i = 0$

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or 1 are respectively $1 - t_i(\mathbf{x})$ and $t_i(\mathbf{x})$. If for some function $t(x)$, $t_i(\mathbf{x}) = t(x_i)$ for $i = 1, \cdots, n$, then $\mathbf{t}$ will be called *simple* and will be denoted by t.

We assume the practical background of the problem provides two positive constants,

a, the loss incurred in deciding "$\theta_i = 0$" when the true value of $\theta_i = 1$,

b, the loss incurred in deciding "$\theta_i = 1$" when the true value of $\theta_i = 0$.

For any Borel set B we define

$$\mu_\theta(B) = \int_B dF(x, \theta), \qquad \theta = 0, 1; \tag{2.1}$$

$$\mu(B) = \mu_0(B) + \mu_1(B). \tag{2.2}$$

Since μ_0 and μ_1 are absolutely continuous with respect to the finite measure μ, generalized probability density functions $f(x, 0)$ and $f(x, 1)$ exist such that for any Borel set B

$$\mu_\theta(B) = \int_B f(x, \theta)\, d\mu, \qquad \text{for} \quad \theta = 0, 1.$$

We note that the relation

$$f(x, 0) + f(x, 1) = 1, \qquad \text{a.e. } (\mu), \tag{2.3}$$

is obtainable from the identity in Borel B,

$$\int_B 1\, d\mu = \mu(B) = \mu_0(B) + \mu_1(B) = \int_B (f(x, 0) + f(x, 1))\, d\mu.$$

The joint generalized probability density function of $\mathbf{x}$ with respect to the product measure μ^n, when the parameter vector is $\boldsymbol{\theta} = (\theta_1, \cdots \theta_n)$, is $f(\mathbf{x}, \boldsymbol{\theta}) = \prod_1^n f(x_i, \theta_i)$. The expected loss on the ith decision in using a decision function $\mathbf{t} = (t_1(\mathbf{x}), \cdots, t_n(\mathbf{x}))$ is, for $i = 1, \cdots, n$,

$$R_i(\mathbf{t}, \boldsymbol{\theta}) = \int [a\theta_i(1 - t_i(\mathbf{x})) + b(1 - \theta_i)t_i(\mathbf{x})]f(\mathbf{x}, \boldsymbol{\theta})\, d\mu^n.$$

The average expected loss on all n decisions, which we define to be the risk of $\mathbf{t}$, is therefore

$$R(\mathbf{t}, \boldsymbol{\theta}) = \frac{1}{n}\sum_1^n R_i(\mathbf{t}, \boldsymbol{\theta}) = \int \frac{1}{n}\sum_1^n \{a\theta_i[1 - t_i(\mathbf{x})] + b(1 - \theta_i)t_i(\mathbf{x})\}\, f(\mathbf{x}, \boldsymbol{\theta})\, d\mu^n. \tag{2.4}$$

This is equivalent to defining the loss of the decision $\mathbf{d} = (d_1, \cdots, d_n)$ in Ω, given $\boldsymbol{\theta}$ in Ω, to be

$$W(\mathbf{d}, \boldsymbol{\theta}) = \frac{1}{n}\sum_1^n [a\theta_i(1 - d_i) + b(1 - \theta_i)\, d_i],$$

since this definition implies that the decision function $\mathbf{t}$ induces the conditional (for fixed $\mathbf{x}$) expected loss

$$W(\mathbf{t}(\mathbf{x}), \boldsymbol{\theta}) = \frac{1}{n}\sum_1^n \{a\theta_i[1 - t_i(\mathbf{x})] + b(1 - \theta_i)t_i(\mathbf{x})\}. \tag{2.5}$$

By the remark (2.3), $f(\mathbf{x}, \boldsymbol{\theta})$ is expressible as

$$(2.6)\quad f(\mathbf{x}, \boldsymbol{\theta}) = \prod_1^n f(x_i, \theta_i) = \prod_1^n [\theta_i f(x_i, 1) + (1 - \theta_i)(1 - f(x_i, 1))].$$

The Halmos-Savage [2] form of the theorem linking sufficient statistics and density factorization shows that $(f(x_1, 1), \cdots, f(x_n, 1))$ is a sufficient statistic for $\boldsymbol{\theta}$ in Ω. Let

$$(2.7)\quad \begin{cases} \mathbf{z} = (z_1, \cdots, z_n), & z_i = f(x_i, 1); \\ \nu_\theta(I) = \mu_\theta(x \mid f(x, 1) \text{ in } I), & \text{for all Borel sets } I; \quad \theta = 0, 1; \\ \nu(I) = \nu_0(I) + \nu_1(I). \end{cases}$$

It is easily verified (see [3] Sec. 32) that the sufficient statistic $\mathbf{z}$ (and, consequently, the measures ν_0, ν_1, and ν) is independent of the choice (2.2) of an underlying measure μ relative to which μ_0 and μ_1 are absolutely continuous. We note that $\mathbf{z}$ has, with respect to the product measure ν^n, the generalized probability density

$$(2.8)\quad d(\mathbf{z}, \boldsymbol{\theta}) = \prod_1^n [\theta_i z_i + (1 - \theta_i)(1 - z_i)].$$

Returning to (2.4) we see that the vector of conditional expectations

$$E[\mathbf{t}(\mathbf{x}) \mid \mathbf{z}] = (E[t_1(\mathbf{x}) \mid \mathbf{z}], \cdots, E[t_n(\mathbf{x}) \mid \mathbf{z}])$$

is a decision rule having the same risk as $\mathbf{t}$. We denote this rule by $\mathbf{t}(\mathbf{z}) = (t_1(\mathbf{z}), \cdots, t_n(\mathbf{z}))$ and, using (2.5) express its risk,

$$R(\mathbf{t}, \boldsymbol{\theta}) = \int W(\mathbf{t}(\mathbf{z}), \boldsymbol{\theta})\, d(\mathbf{z}, \boldsymbol{\theta})\, d\nu^n$$

$$(2.9)\quad = \frac{a}{n} \sum_1^n \theta_i \int (1 - t_i(\mathbf{z}))\, d(\mathbf{z}, \boldsymbol{\theta})\, d\nu^n + \frac{b}{n} \sum_1^n (1 - \theta_i) \cdot \int t_i(\mathbf{z})\, d(\mathbf{z}, \boldsymbol{\theta})\, d\nu^n.$$

3. Simple decision functions. If $\mathbf{t} = t$ is a simple decision function, then (2.9) simplifies to

$$(3.1)\quad R(t, \boldsymbol{\theta}) = \frac{a}{n} \sum_1^n \theta_i \int (1 - t(z))z\, d\nu + \frac{b}{n} \sum_1^n (1 - \theta_i) \int t(z)(1 - z)\, d\nu.$$

Setting $\bar{\theta} = (1/n)\sum \theta_i =$ proportion of 1's among the n components of θ, we can write (3.1) in the form

$$(3.2)\quad R(t, \boldsymbol{\theta}) = \int a\bar{\theta}(1 - t(z))z + b(1 - \bar{\theta})t(z)(1 - z)\, d\nu.$$

The value $\bar{\theta}$ is necessarily rational, of the form k/n for $k = 0, \cdots, n$. We now define for any real number p such that $0 \leqq p \leqq 1$, and any ν-measurable function $t(z)$ such that $0 \leqq t(z) \leqq 1$, the expression

$$B(t, p) = \int ap(1 - t(z))z + b(1 - p)t(z)(1 - z)\, d\nu. \tag{3.3}$$

Thus (3.2) becomes

$$R(t, \theta) = B(t, \bar{\theta}). \tag{3.4}$$

From (3.3) it is clear that for any fixed p, $B(t, p)$ is a minimum if and only if for almost every $(\nu)z$, $t(z)$ is of the form

$$t_p(z) = \begin{cases} 1, & \text{if } apz > b(1 - p)(1 - z); \\ 0, & \text{if } apz < b(1 - p)(1 - z); \\ \text{arbitrary}, & \text{if } apz = b(1 - p)(1 - z). \end{cases} \tag{3.5}$$

The minimum value of $B(t, p)$ is

$$\begin{aligned} \phi(p) = B(t_p, p) &= \int \min\,[apz, b(1 - p)(1 - z)]\, d\nu \\ &= ap \int_0^{C(p)} d\nu_1 + b(1 - p)\left(1 - \int_0^{C(p)} d\nu_0\right) \end{aligned} \tag{3.6}$$

where $C(p) = b(1 - p) / [ap + b(1 - p)]$ for $0 \leqq p \leqq 1$. It follows from (3.4) that, for any simple decision function t and any θ in Ω,

$$R(t, \theta) \geqq \phi(\bar{\theta}), \tag{3.7}$$

equality holding if and only if $t(z)$ is of the form (3.5) with $p = \bar{\theta}$.

We shall now establish some properties of the function $\phi(p)$ defined for $0 \leqq p \leqq 1$ by (3.6). For $0 \leqq p_1 < p_2 \leqq 1$, and $0 < s < 1$,

$$\begin{aligned} & s \min[ap_1z, \quad b(1 - p_1)(1 - z)] + (1 - s) \min[ap_2z, \quad b(1 - p_2)(1 - z)] \\ &\quad = \min[asp_1z, \quad bs(1 - p_1)(1 - z)] \\ &\qquad\qquad + \min[a(1 - s)p_2z, \quad b(1 - s)(1 - p_2)(1 - z)] \\ &\quad \leqq \min[a\{sp_1 + (1 - s)p_2\}z, \quad b\{1 - sp_1 - (1 - s)p_2\}(1 - z)], \end{aligned}$$

with strict inequality if and only if $p_1 < C(z) < p_2$ (or alternatively $C(p_2) < z < C(p_1)$). Integrating with respect to ν we obtain from (3.6) that ϕ is a concave function of p,

$$s\phi(p_1) + (1 - s)\,\phi(p_2) \leqq \phi(sp_1 + (1 - s)p_2), \tag{3.8}$$

with equality if and only if $\nu[z \mid C(p_2) < z < C(p_1)] = 0$. Since $\phi(0) = \phi(1) = 0$, this implies

$$\phi(p) > 0, \qquad 0 < p < 1, \qquad \text{unless } \nu[z \mid 0 < z < 1] = 0. \tag{3.9}$$

The exception here is the trivial case where ν_0 and ν_1 (and hence μ_0 and μ_1) are disjoint measures.

The continuity of $\phi(p)$ can be established in the following form. If $0 \leqq p < p' \leqq 1$, then $0 \leqq C(p') < C(p) \leqq 1$ and

$$\min[azp,\ b(1-z)(1-p)] - \min[azp',\ b(1-z)(1-p')]$$

$$= \begin{cases} -az(p'-p), & 0 \leqq z < C(p'); \\ b(1-z)(1-p) - azp', & C(p') \leqq z \leqq C(p); \\ b(1-z)(p'-p), & C(p) < z \leqq 1. \end{cases}$$

Hence it follows that, for $0 \leqq z \leqq 1$,

$$-az(p'-p) \leqq \min[azp,\ b(1-z)(1-p)] - \min[azp',\ b(1-z)(1-p')] \leqq b(1-z)(p'-p).$$

Integrating with respect to ν, we obtain

$$-a(p'-p) \leqq \phi(p) - \phi(p') \leqq b(p'-p), \qquad p' > p. \tag{3.10}$$

Interchanging p and p', multiplying by -1 achieves

$$-b(p-p') \leqq \phi(p) - \phi(p') \leqq a(p-p'), \qquad p' < p. \tag{3.11}$$

From (3.3) we have, for any $0 \leqq p, p' \leqq 1$,

$$B(t_p, p') - \phi(p) = \int (p'-p)\{a(1-t_p(z))z - bt_p(z)(1-z)\}\, d\nu$$

$$= (p'-p) \int \{a(1-t_p(z))z - bt_p(z)(1-z)\}\, d\nu, \tag{3.12}$$

$$\leqq \begin{cases} (p'-p)a, & p' > p; \\ (p-p')b, & p' < p. \end{cases}$$

The last three inequalities, (3.10) to (3.12), and the definition of ϕ imply

$$0 \leqq B(t_p, p') - \phi(p') \leqq (a+b)\,|\,p-p'\,|.$$

From (3.4) it follows that for any $0 \leqq p \leqq 1$ and any θ in Ω,

$$0 \leqq R(t_p, \theta) - \phi(\bar{\theta}) \leqq (a+b)\,|\,p-\bar{\theta}\,|.$$

Thus, we have proved

THEOREM 1. *Suppose that in some manner an approximate value p of the true proportion $\bar{\theta}$ is known. Then the statistician who uses the simple decision function $\mathbf{t} = t_p$ will achieve a risk $R(t_p, \theta)$ which is within $(a+b)\,|\,p-\bar{\theta}\,|$ of the minimum attainable risk $\phi(\bar{\theta})$ in the class of all simple decision functions.*

4. "Consistent" estimation of a proportion. We use (without loss of generality) the canonical form of the problem (see (2.6) to (2.9)), assuming only that the

original measures, μ_0 and μ_1, are not identical. Thus we are concerned with the random vector $\mathbf{z} = (z_1, \cdots, z_n)$ having the density $d(\mathbf{z}, \boldsymbol{\theta})$ defined by (2.8). We consider the problem of finding an estimator $p_n(\mathbf{z})$ for the proportion $\bar{\theta}$ of 1's in the first n coordinates of $\boldsymbol{\theta}$.

Since our principal interest is in the asymptotic estimation problem, we will consider the sequence of problems defined for $n = 1, 2, \cdots$ as embedded in the probability space of infinite sequences $\mathbf{z} = (z_1, z_2, \cdots)$ with p-measure induced by the density $d(\mathbf{z}, \boldsymbol{\theta})$ defined on the first n coordinates of $\mathbf{z}$ and $\boldsymbol{\theta}$, respectively. We emphasize this aspect in this and the following section by referring to $\boldsymbol{\theta}$ in Ω_∞.

Avoiding a discussion of "optimum" estimation, we devote this section to the consideration of a subclass H of the class U of all unbiased estimators of $\bar{\theta}$. This subclass H is to be the class of all estimators of the form

$$\bar{h}(\mathbf{z}) = \frac{1}{n}\sum_1^n h(z_i), \tag{4.1}$$

where $h(z)$ is an unbiased estimator of θ.

As a measure of the risk of an estimator in H we use its variance,

$$n \operatorname{Var} \bar{h}(\mathbf{z}) = \bar{\theta}\, V_1(h) + (1 - \bar{\theta})\, V_0(h), \tag{4.2}$$

where $V_1(h) = \int (h(z) - 1)^2\, d\nu_1$ and $V_0(h) = \int (h(z))^2\, d\nu_0$. We single out an interesting subclass of H by investigating the existence and representation of elements h minimizing, for fixed p, with $0 < p < 1$,

$$pV_1(h) + (1 - p)V_0(h). \tag{4.3}$$

For any pair of real numbers $\lambda = (\lambda_0, \lambda_1)$ we define an extension of (4.3) to the set of all $g(z)$ such that $\int |g(z)|\, d\nu < \infty$:

$$F_\lambda(g) = p\left[\int g^2\, d\nu_1 - 1\right] + (1 - p)\int g^2\, d\nu_0 - 2\lambda_1\left[\int g\, d\nu_1 - 1\right] - 2\lambda_0 \int g\, d\nu_0. \tag{4.4}$$

Using the density representation of these integrals and the restriction on the domain of g, we have

$$F_\lambda(g) = \int g^2(z)[pz + (1 - p)(1 - z)] - 2g(z)[\lambda_1 z + \lambda_0(1 - z)] \cdot d\nu - p + 2\lambda_1. \tag{4.5}$$

For fixed λ_0 and λ_1 the integrand here is a minimum, for each fixed z, if and only if

$$g(z) = g_p(z \mid \lambda) = \frac{\lambda_1 z + \lambda_0(1 - z)}{pz + (1 - p)(1 - z)}, \qquad \text{a. e. } (\nu). \tag{4.6}$$

Since $g_p(z \mid \lambda) \leqq \max [|\lambda_1|, |\lambda_0|] / \min [p, 1 - p]$, we find that $g_p(z \mid \lambda)$ is a unique (ν) minimum of $F_\lambda(g)$ over its domain. If there exists a determination of λ_0 and λ_1 such that $g_p(z \mid \lambda)$ is an unbiased estimator of θ, we will denote it by $h_p(z)$. Then we have for all unbiased h

$$pV_1(h) + (1 - p)V_0(h) = F_\lambda(h) \geqq F_\lambda(h_p) = pV_1(h_p) + (1 - p)V_0(h_p), \tag{4.7}$$

where equality holds only if $h(z) \equiv h_p(z) \quad (\nu)$.

The estimator $g_\lambda(z \mid p)$ will be unbiased if λ_0 and λ_1 satisfy

$$\lambda_1 \int \frac{z}{pz + (1 - p)(1 - z)}\, d\nu_i + \lambda_0 \int \frac{1 - z}{pz + (1 - p)(1 - z)}\, d\nu_i = i, \qquad i = 0, 1. \tag{4.8}$$

The determinant of these equations is

$$\left[\int \frac{z(1 - z)}{pz + (1 - p)(1 - z)}\, d\nu\right]^2 - \int \frac{(1 - z)^2}{pz + (1 - p)(1 - z)}\, d\nu \cdot \int \frac{z^2}{pz + (1 - p)(1 - z)}\, d\nu.$$

It is nonpositive by the Schwarz inequality. If it is equal to zero, z and $1 - z$ are linearly dependent a.e. (ν). Since z and $1 - z$ are densities, linear dependence would imply that ν_0 and ν_1 are identical; this possibility has been excluded here.

Thus, an explicit representation of estimators in H, minimizing (4.2) for θ such that $\bar{\theta} = p$, is obtained as

$$\bar{h}_p(\mathbf{z}) = \frac{1}{n} \sum_1^n h_p(z_i), \qquad 0 < p < 1,$$

where $h_p(z) = g_p(z \mid \lambda)$, and λ is a solution of (4.8). This class of estimators merits our interest since each is admissible relative to H and each satisfies the following theorem.

THEOREM 2. *Let $\bar{h}(\mathbf{z})$ be any estimator in H such that $|h(z)| + 1 < M < \infty$. Define*

$$p_n(\mathbf{z}) = \begin{cases} 0, & \bar{h}(\mathbf{z}) < 0; \\ \bar{h}(\mathbf{z}), & 0 \leqq \bar{h}(\mathbf{z}) \leqq 1; \\ 1, & 1 < \bar{h}(\mathbf{z}). \end{cases}$$

Then, (a) $0 \leqq p_n(z) \leqq 1$, and (b) for any $\epsilon > 0$ there exists an $N(\epsilon)$ such that, for any θ in Ω_∞,

$$\Pr[|p_n(z) - \bar{\theta}| > \epsilon \quad \text{for some } n \geqq N(\epsilon)] \leqq \epsilon.$$

PROOF. It will clearly be sufficient to show that $\bar{h}(\mathbf{z})$ satisfies part (b). For this purpose we introduce $\bar{h}_i(k)$ for $i = 0$ or 1 and $k = 1, 2, \cdots$ to denote the arithmetic mean of k independent random variables, each having the probability measure $\nu_i h^{-1}$. We express

$$\bar{h}(\mathbf{z}) = (1/n)\sum[\theta_j h(z_j) + (1 - \theta_j)h(z_j)]$$

$$= \bar{\theta} \sum \theta_j h(z_j) / \sum \theta_j + (1 - \bar{\theta}) \sum (1 - \theta_j) h(z_j) / \sum (1 - \theta_j) \tag{4.9}$$

$$= \bar{\theta} \bar{h}_1(n\bar{\theta}) + (1 - \bar{\theta}) \bar{h}_0(n - n\bar{\theta}),$$

$$\bar{h}(\mathbf{z}) - \bar{\theta} = \bar{\theta}[\bar{h}_1(n\bar{\theta}) - 1] + (1 - \bar{\theta}) \bar{h}_0(n - n\bar{\theta}). \tag{4.10}$$

The strong law of large numbers yields the existence of functions $N_1(\eta)$ and $N_0(\eta)$, defined for $\eta > 0$ and such that

$$\Pr[|\bar{h}_i(k) - i| > \eta \quad \text{for some } k \geqq N_i(\eta)] \leqq \eta, \qquad i = 0, 1. \tag{4.11}$$

We fix $\epsilon > 0$ and consider the term $\bar{\theta}[\bar{h}_1(n\bar{\theta}) - 1]$ of (4.10). Since $\bar{\theta} \leqq 1$ and $|\bar{h}_1(n\bar{\theta}) - 1| \leqq M$, we have

$$|\bar{\theta}[\bar{h}_1(n\bar{\theta}) - 1]| \leqq \min \{\bar{\theta}M, \ |\bar{h}_1(n\bar{\theta}) - 1|\}. \tag{4.12}$$

Hence, for any fixed integer N and any fixed point $\boldsymbol{\theta}$ in Ω_∞, if $\mathbf{z}$ is such that there exist $n_{\mathbf{z}} \geqq N$ with $\bar{\theta}[\bar{h}_1(n_{\mathbf{z}}\bar{\theta}) - 1] > \epsilon/2$, then

$$n_{\mathbf{z}}\bar{\theta} > n_{\mathbf{z}}M^{-1}\epsilon/2, \qquad |\bar{h}_1(n_{\mathbf{z}}\bar{\theta}) - 1| > \epsilon/2.$$

Hence there exists $k > NM^{-1}\epsilon/2$ such that $|\bar{h}_1(k) - 1| > \epsilon/2$. Consequently

$$\begin{aligned} &\Pr\{\bar{\theta}[\bar{h}_1(n\bar{\theta}) - 1] > \epsilon/2 \text{ for some } n \geqq N\} \\ &\qquad \leqq \Pr\{|\bar{h}_1(k) - 1| > \epsilon/2 \text{ for some } k > NM^{-1}\epsilon/2\}. \end{aligned} \tag{4.13}$$

Thus, if $N \geqq 2M\epsilon^{-1}N_1(\epsilon/2)$, it follows from (4.11) for $i = 1$ that the right side of (4.13) is less than $\epsilon/2$ uniformly for all $\boldsymbol{\theta}$ in Ω_∞.

We can deal in a similar fashion with the term $(1 - \bar{\theta})\bar{h}_0(n - n\bar{\theta})$ of (4.10). Thus we obtain that part (b) is satisfied by

$$N(\epsilon) = 2M\epsilon^{-1} \max [N_1(\epsilon/2), N_0(\epsilon/2)]. \tag{4.14}$$

5. Nonsimple decision functions t. We have seen in Section 3 that if p is a good approximation to $\bar{\theta}$, then the simple decision function $\mathbf{t} = t_p$ (see (3.5)) does about as well as is possible for any simple decision function. Although a good approximation p to $\bar{\theta}$ is not generally available to the statistician, we have seen in Section 4 that for large n a good estimator $p_n(\mathbf{z})$ of $\bar{\theta}$ is always available.

It is natural to combine these two results by using the decision function t_p with the constant p replaced by the random variable $p_n(z)$. This amounts to using the nonsimple decision function $t^* = (t_1^*(\mathbf{z}), \cdots, t_n^*(\mathbf{z}))$ such that for $i = 1, \cdots, n$,

$$t_i^*(\mathbf{z}) = \begin{cases} 1, & \text{if } z_i > C(p_n(\mathbf{z})), \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

(We have chosen, arbitrarily, one way of resolving the ambiguity in the definition (3.5) when $z = C(p)$.)

In practice, $\mathbf{t}^*$ can be used only when all the values $z_1, \cdots, z_n$ are known before the individual decisions on the values of $\theta_1, \cdots, \theta_n$ have to be made.

We shall now investigate the behavior, for large n, of the risk function $R(\mathbf{t}^*, \boldsymbol{\theta})$.

We begin by considering the loss of the decision rule t_p (determined by (3.5) and by $t_p(C(p)) = 0$), where $\mathbf{z}$ is observed and $\boldsymbol{\theta}$ is the true parameter point. From (2.5) we obtain

$$\begin{aligned} W(t_p(\mathbf{z}), \boldsymbol{\theta}) &= a\bar{\theta}[\textstyle\sum\theta_i(1 - t_p(z_i)) / \sum\theta_i] \\ &\quad + b(1 - \bar{\theta})[\textstyle\sum(1 - \theta_i)t_p(z_i) / \sum(1 - \theta_i)] \\ &= a\bar{\theta}S_1(C(p) \mid n\bar{\theta}) + b(1 - \bar{\theta})[1 - S_0(C(p) \mid n - n\bar{\theta})], \end{aligned} \tag{5.2}$$

where $S_i(v \mid k)$ is the sample distribution function of k independent random variables, each distributed with probability measure ν_i for $i = 0$ or 1 and $k = 1, 2, \cdots$. For future use we define

$$D_1^+(k) = \sup_{0 \le v \le 1}\left[S_1(v \mid k) - \int_0^v d\nu_1\right], \quad D_0^-(k) = \sup_{0 \le v \le 1}\left[\int_0^v d\nu_0 - S_0(v \mid k)\right]. \tag{5.3}$$

Using an alternative form of (3.3), it follows from (5.2) that

$$\begin{aligned} W(t_p(\mathbf{z}), \boldsymbol{\theta}) &= a\bar{\theta}\left\{S_1(C(p) \mid n\bar{\theta}) - \int_0^{C(p)} d\nu_1\right\} \\ &\quad + b(1 - \bar{\theta})\left\{\int_0^{C(p)} d\nu_0 - S_0(C(p) \mid n - n\bar{\theta})\right\} + B(t_p, \boldsymbol{\theta}). \end{aligned}$$

Hence, from (5.3) and Theorem 1,

$$W(t_p(\mathbf{z}), \boldsymbol{\theta}) \le a\bar{\theta}D_1^+(n\bar{\theta}) + b(1 - \bar{\theta})D_0^-(n - n\bar{\theta}) + \phi(\bar{\theta}) + (a + b)\,|p - \bar{\theta}|\,.$$

Since this holds for each $0 \le p \le 1$ we have, for any estimate $p_n(\mathbf{z})$ satisfying (a) of Theorem 1,

$$\begin{aligned} W(t_{p_n(\mathbf{z})}(\mathbf{z}), \boldsymbol{\theta}) &\le a\bar{\theta}D_1^+(n\bar{\theta}) \\ &\quad + b(1 - \bar{\theta})D_0^-(n - n\bar{\theta}) + \phi(\bar{\theta}) + (a + b)\,|p_n(z) - \bar{\theta}|\,. \end{aligned} \tag{5.4}$$

By the Glivenko-Cantelli Theorem [4]

$$\Pr\left[\lim_{k=\infty} D_1^+(k) = 0\right] = \Pr\left[\lim_{k=\infty} D_0^-(k) = 0\right] = 1.$$

This is equivalent to the existence of functions $N_1^*(\eta)$ and $N_0^*(\eta)$, defined for $\eta > 0$, such that

$$\Pr[D_1^+(k) > \eta \text{ for some } k \ge N_1^*(\eta)] \le \eta,$$

$$\Pr[D_0^-(k) > \eta \text{ for some } k \ge N_0^*(\eta)] \le \eta.$$

As in (4.11) to (4.14), we have that if

$$N^*(\epsilon) = 2\epsilon^{-1} \max[aN_1^*(\epsilon/2), bN_0^*(\epsilon/2)], \qquad \epsilon > 0,$$

then, for any $\boldsymbol{\theta}$ in Ω_∞,

$$\Pr[a\bar{\theta}D_1^+(n\bar{\theta}) + b(1 - \bar{\theta})D_0^-(n - n\bar{\theta}) > \epsilon \text{ for some } n \ge N^*(\epsilon)] \le \epsilon. \tag{5.5}$$

Returning to (5.4) we see that, for any $p_n(\mathbf{z})$ satisfying the conclusion of Theorem 2, (5.5) and Theorem 2 combine to furnish for any $\epsilon > 0$

$$n_0(\epsilon) = \max[N^*(\epsilon/2), \quad N(\epsilon/2(a + b))]$$

such that, for any $\boldsymbol{\theta}$ in Ω_∞,

$$\Pr\{W(t_{p_n(\mathbf{z})}(\mathbf{z}), \boldsymbol{\theta}) - \phi(\bar{\theta}) > \epsilon \text{ for some } n \geqq n_0(\epsilon)\} \leqq \epsilon. \tag{5.6}$$

The argument from (5.2) onward proves

THEOREM 3. *If $p_n(\mathbf{z})$ satifies the conclusions (a) and (b) of Theorem 2, then the decision function $\mathbf{t}^*$ defined by (5.1) is such that to any $\epsilon > 0$ there corresponds an $n_0(\epsilon)$ such that for any $\boldsymbol{\theta}$ in Ω_∞*

$$\Pr\{W(\mathbf{t}^*(\mathbf{z}), \boldsymbol{\theta}) \leqq \phi(\bar{\theta}) + \epsilon \text{ for all } n \geqq n_0(\epsilon)\} \geqq 1 - \epsilon.$$

Since $W(\mathbf{t}^*(\mathbf{z}), \boldsymbol{\theta}) \leqq \max[a, b]$, Theorem 3 implies

THEOREM 4. *Under the assumptions of Theorem 3, $\mathbf{t}^*$ is such that to any $\epsilon > 0$ there corresponds an $n_1(\epsilon)$ such that for any $n \geqq n_1(\epsilon)$ and any $\boldsymbol{\theta}$ in Ω,*

$$R(\mathbf{t}^*, \boldsymbol{\theta}) < \phi(\bar{\theta}) + \epsilon.$$

Thus, $\mathbf{t}^*$ is a nonsimple decision function which for large n does about as well as could be done by any simple decision function even if $\bar{\theta}$ were known exactly.

6. Invariant and R-invariant decision functions. Let $(P(1), \cdots, P(n))$ be an arbitrary permutation of $(1, \cdots, n)$. Define, for any real vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$,

$$P\boldsymbol{\xi} = (\xi_{P(1)}, \cdots, \xi_{P(n)}). \tag{6.1}$$

From (2.5) and (2.8) we note that $W(P\mathbf{t}, P\boldsymbol{\theta}) = W(\mathbf{t}, \boldsymbol{\theta})$ and $d(P\mathbf{z}, P\boldsymbol{\theta}) = d(\mathbf{z}, \boldsymbol{\theta})$. From (2.9) we obtain for any decision function $\mathbf{t}$, by a change of variable of integration,

$$\begin{aligned} R(\mathbf{t}, P\boldsymbol{\theta}) &= \int W(\mathbf{t}(\mathbf{z}), P\boldsymbol{\theta})\, d(\mathbf{z}, P\boldsymbol{\theta})\, d\nu^n \\ &= \int W(\mathbf{t}(P\mathbf{z}), P\boldsymbol{\theta})\, d(P\mathbf{z}, P\boldsymbol{\theta})\, d\nu^n = R(P^{-1}\mathbf{t}P, \boldsymbol{\theta}). \end{aligned} \tag{6.2}$$

We call a decision function $\mathbf{t}$ *invariant* if $P^{-1}\mathbf{t}P = \mathbf{t}$ for all P. We denote by *R-invariant* a decision function $\mathbf{t}$ such that

$$R(P^{-1}\mathbf{t}P, \boldsymbol{\theta}) = R(\mathbf{t}, \boldsymbol{\theta}), \qquad \text{for all } P \text{ and } \boldsymbol{\theta}. \tag{6.3}$$

The risk of an R-invariant decision function can be expressed as an explicit function of $\bar{\theta}$ by formally averaging the representation (2.9) over all P. From (2.9), (6.2), and (6.3),

$$R(\mathbf{t}, \boldsymbol{\theta}) = \frac{1}{n!} \sum_P R(\mathbf{t}, P\boldsymbol{\theta}) = \frac{1}{n!} \sum_P \int W(\mathbf{t}(\mathbf{z}), P\boldsymbol{\theta})\, d(\mathbf{z}, P\boldsymbol{\theta})\, d\nu^n. \tag{6.4}$$

Changing the order of summations and integration, we find that if $\mathbf{t}$ is R-invariant its risk, $R(\mathbf{t}, \boldsymbol{\theta})$, is expressible in the form

$$R(\mathbf{t}, \boldsymbol{\theta}) = \frac{1}{n}\sum_1^n \int \left\{ a\,\frac{1 - t_i(\mathbf{z})}{n!} \sum_P \theta_{P(i)}\, d(\mathbf{z}, P\boldsymbol{\theta}) + b\,\frac{t_i(\mathbf{z})}{n!} \sum_P (1 - \theta_{P(i)})\, d(\mathbf{z}, P\boldsymbol{\theta}) \right\} dv^n. \tag{6.5}$$

For any real vector $\mathbf{r} = (r_1, \cdots, r_n)$ we define for every integer k

$$L(k, n, \mathbf{r}) = \begin{cases} \dfrac{1}{n!} \displaystyle\sum_P \prod_1^k r_{P(j)} \prod_{k+1}^n (1 - r_{P(j)}), & k = 0, 1, \cdots, n; \\ 0, & \text{otherwise}; \end{cases}$$

$$L(k, n-1, \check{r}_i = L[k, n-1, (r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n)], \quad i = 1, 2, \cdots, n.$$

Then

$$\begin{aligned} \sum_P \theta_{P(i)}\, d(\mathbf{z}, P\boldsymbol{\theta}) &= z_i \sum_{\theta_{P(i)}=1} \prod_{j \neq i} [\theta_{P(j)} z_j + (1 - \theta_{P(j)})(1 - z_j)] \\ &= z_i \left(\sum_1^n \theta_k \right) (n-1)!\, L\left(\sum_1^n \theta_k - 1, \quad n-1, \quad \check{z}_i \right); \end{aligned} \tag{6.6}$$

$$\begin{aligned} \sum_P (1 - \theta_{P(i)})\, d(\mathbf{z}, P\boldsymbol{\theta}) &= (1 - z_i) \sum_1^n (1 - \theta_k)(n-1)! \\ &\quad \cdot L\left(\sum_1^n \theta_k, \quad n-1, \quad \check{z}_i \right). \end{aligned} \tag{6.7}$$

It follows from (6.5) that for R-invariant $\mathbf{t}$

$$\begin{aligned} R(\mathbf{t}, \boldsymbol{\theta}) = \frac{1}{n}\sum_1^n \int \{ & a[1 - t_i(\mathbf{z})] z_i \bar{\theta} L(n\bar{\theta} - 1, \quad n-1, \quad \check{z}_i) \\ & + b t_i(\mathbf{z})(1 - z_i)(1 - \bar{\theta}) L(n\bar{\theta}, \quad n-1, \check{z}_i) \}\, dv^n. \end{aligned} \tag{6.8}$$

Denoting the ith summand of (6.8) by $R_i(\mathbf{t} \mid \bar{\theta})$, we note for later use that, if $\mathbf{t}$ is actually invariant,

$$R_i(\mathbf{t} \mid \bar{\theta}) = R_1(\mathbf{t} \mid \bar{\theta}) = R(\mathbf{t}, \boldsymbol{\theta}), \qquad i = 1, \cdots, n. \tag{6.9}$$

This follows from the invariance and a permutation of the variables of integration in the representation of $R_i(\mathbf{t} \mid \bar{\theta})$ in (6.8). It further follows from this representation that, for any fixed $\boldsymbol{\theta}$, the integrand of $R_i(\mathbf{t} \mid \bar{\theta})$ is, for each fixed $\mathbf{z}$, at a minimum with respect to t_i if and only if t_i is of the form defined, for $p = 0, 1/n, \cdots, (n-1)/n, 1$, by

$$t_{ip}(\mathbf{z}) = \left\{ \begin{matrix} 1, \\ 0, \\ \text{arbitrary}, \end{matrix} \right\} \begin{matrix} az_i p L(np - 1, \\ n-1, \quad \check{z}_i) \end{matrix} \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} b(1 - z_i)(1 - p) \cdot L(np, \quad n-1, \quad \check{z}_i), \tag{6.10}$$

with p equal to $\bar{\theta}$. We denote $(t_{1p}(\mathbf{z}), \cdots, t_{np}(\mathbf{z}))$ by $\mathbf{t}_p$. With a proper determination of the arbitrary part, any decision function of this form (6.10) is invariant. Hence, the representation of its risk in the form (6.8) is valid and $R(\mathbf{t}, \boldsymbol{\theta})$ for each fixed $\boldsymbol{\theta}$ is minimized over the class of R-invariant decision functions by an invariant decision function of the form (6.10).

Let

$$\phi_n(\bar{\theta}) = \inf_{R\text{-invariant } \mathbf{t}} R(\mathbf{t}, \boldsymbol{\theta}). \tag{6.11}$$

From (6.8), (6.9), and (6.10) we obtain

$$\begin{aligned}\phi_n(\bar{\theta}) &= R_1(t_{\bar{\theta}} \mid \bar{\theta}) \\ &= \int \min\,[az_1\bar{\theta}L(n\bar{\theta} - 1, \quad n - 1, \check{z}_1), \\ &\qquad b(1 - z_1)(1 - \bar{\theta})L(n\bar{\theta}, \quad n - 1, \quad \check{z}_1)]\, d\nu^n.\end{aligned} \tag{6.12}$$

Since $L(k, \quad n - 1, \quad \check{z}_1)$ is a symmetric function of $z_2, \cdots, z_n$, we have for any symmetric measurable set S

$$\int_S L(k, \quad n - 1, \quad \check{z}_1)\, d\nu^{n-1} = \nu_1^k \nu_0^{n-1-k}(S). \tag{6.13}$$

It follows that the integral with respect to $z_2, \cdots, z_n$ of a z_1-section of the integrand of (6.12) is

$$az_1\bar{\theta}[1 - \nu_1^{n\bar{\theta}-1}\nu_0^{n-n\bar{\theta}}(S_{z_1})] + b(1 - z_1)(1 - \bar{\theta})\, \nu_1^{n\bar{\theta}}\nu_0^{n-1-n\bar{\theta}}(S_{z_1})$$

where S_{z_1} is the symmetric set

$$\begin{aligned}S_{z_1} = [(z_2, \cdots, z_n) \mid\ & az_1\bar{\theta}L(n\bar{\theta} - 1, \quad n - 1, \quad \check{z}_1) \\ & > b(1 - z_1)(1 - \bar{\theta})L(n\bar{\theta}, \quad n - 1, \quad \check{z}_1)].\end{aligned}$$

The developments from (6.4) onward set the stage for

THEOREM 5. *If $\phi(\bar{\theta})$ and $\phi_n(\bar{\theta})$ are defined by* (3.6) *and* (6.12), *respectively, then for any $\epsilon > 0$ there exists $N(\epsilon)$ such that for any $n \geqq N(\epsilon)$ and any $\boldsymbol{\theta}$*

$$\phi(\bar{\theta}) - \epsilon < \phi_n(\bar{\theta}) \leqq \phi(\bar{\theta}).$$

PROOF. That $\phi_n \leqq \phi$ for all $\boldsymbol{\theta}$ follows from the fact that every simple decision function is invariant.

For the nontrivial part of the proof we fix $\epsilon > 0$. From the continuity of $\phi(p)$ ((3.10) and (3.11)) we obtain a $\delta(\epsilon)$ such that

$$\phi(p) - \epsilon < 0 \qquad \text{if } p < \delta \quad \text{or } p > 1 - \delta. \tag{6.14}$$

Thus, it will be sufficient to show the existence of an $N(\epsilon)$ which suffices for the theorem when $0 < \delta \leqq \bar{\theta} \leqq 1 - \delta$. We will obtain this from the following measure theoretic lemma, a slight generalization of which is proved in [5].

LEMMA. *If m_0 and m_1 are nondisjoint p-measures on a σ-algebra of subsets χ of a set X, then for every $\epsilon, \delta > 0$ there exists $N(\epsilon, \delta)$ such that for any pair of positive integers r, s with $r + s \geqq N(\epsilon, \delta)$ and $\delta \leqq r/(r+s) \leqq 1 - \delta$,*

$$|m_0^r m_1^s(S) - m_0^{r-1} m_1^{s+1}(S)| < \epsilon \text{ uniformly for all symmetric } S \text{ in } \chi^{r+s}.$$

From this lemma we obtain that if $n - 1 \geqq N(\eta, \delta^*)$ and $\delta^* \leqq n\bar{\theta}/(n-1) \leqq 1 - \delta^*$,

$$[1 - \nu_1^{n\bar{\theta}-1}\nu_0^{n-n\bar{\theta}}(S_{z_1})] + \nu_1^{n\bar{\theta}}\nu_0^{n-1-n\bar{\theta}}(S_{z_1}) > 1 - \eta$$

uniformly for all z_1. From this it follows that $\phi_n(\bar{\theta}) > (1-\eta)\phi(\bar{\theta}) = \phi(\bar{\theta}) - \eta\phi(\bar{\theta})$. Since the $\max_p \phi(p) \leqq c = ab/(a+b)$, we see that in view of (6.14) the choice $N(\epsilon) = N(c^{-1}\epsilon, 2^{-1}\delta(\epsilon))$ will complete the proof of Theorem 5.

Theorem 5 can be combined with Theorem 3 or Theorem 4 to give a strengthened endorsement of $\mathbf{t}^*$. For large n, $\mathbf{t}^*$ does about as well as could be done by any R-invariant decision function even if $\bar{\theta}$ were known exactly.

7. Bayes' and minimax solutions. Let $\mathbf{t} = (t_1(\mathbf{z}), \cdots, t_n(\mathbf{z}))$ be any decision function in the compound decision problem. For any $\boldsymbol{\theta}$ in Ω we can write the risk (2.9) in the form

$$R(\mathbf{t}, \boldsymbol{\theta}) = \int \frac{1}{n} \sum_1^n \{a\theta_i[1 - t_i(\mathbf{z})]\, d(\mathbf{z}, \boldsymbol{\theta}) + b(1-\theta_i)t_i(\mathbf{z})\, d(\mathbf{z}, \boldsymbol{\theta})\}\, d\nu^n.$$

By a weight function we mean any function $\beta(\boldsymbol{\theta}) \geqq 0$ defined on Ω and not identically 0. For any weight function $\beta(\boldsymbol{\theta})$ and any decision function $\mathbf{t}$ we define the weighted risk of $\mathbf{t}$ relative to $\beta(\boldsymbol{\theta})$ as

$$B(\mathbf{t}, \beta) = \sum_{\boldsymbol{\omega}} \beta(\boldsymbol{\omega}) \cdot R(\mathbf{t}, \boldsymbol{\omega}) = \int \frac{1}{n} \sum_1^n \{(1 - t_i(\mathbf{z}))\, a \sum_{\boldsymbol{\omega}} \beta(\boldsymbol{\omega})\omega_i\, d(\mathbf{z}, \boldsymbol{\omega}) + t_i(\mathbf{z}) b \sum_{\boldsymbol{\omega}} \beta(\boldsymbol{\omega})(1-\omega_i)\, d(\mathbf{z}, \boldsymbol{\omega})\}\, d\nu^n.$$

For fixed $\beta(\boldsymbol{\theta})$ this will be a minimum if and only if for almost every $(\nu^n)\mathbf{z}$ and for every $i = 1, \cdots, n$, $t_i(\mathbf{z})$ is of the form

$$(7.1) \qquad t_i(\mathbf{z} \mid \beta) = \begin{Bmatrix} 1, \\ 0, \\ \text{arbitrary}, \end{Bmatrix} a \sum_{\boldsymbol{\omega}} \beta(\boldsymbol{\omega})\omega_i\, d(\mathbf{z}, \boldsymbol{\omega}) \begin{Bmatrix} > \\ < \\ = \end{Bmatrix} b \sum_{\boldsymbol{\omega}} \beta(\boldsymbol{\omega})(1-\omega_i)\, d(\mathbf{z}, \boldsymbol{\omega}).$$

Any decision function $\mathbf{t}$ of the form (7.1) is called a *Bayes' solution* relative to the weight function $\beta(\boldsymbol{\theta})$.

For the remainder of this section we restrict our attention to symmetric $\beta(\boldsymbol{\theta})$, that is, such that

$$(7.2) \qquad \binom{n}{k} \beta(\boldsymbol{\theta}) = \beta_k \qquad \text{for all } \boldsymbol{\theta} \text{ with } \sum_1^n \theta_j = k, \qquad k = 0, 1, \cdots, n.$$

For symmetric β we obtain the representation (see (6.8) to (6.11))

$$a \sum_{\omega} \beta(\omega)\omega_i \, d(\mathbf{z}, \omega) = az_i \sum_{k=0}^{n} \beta_k \frac{k}{n} L(k-1, \quad n-1, \quad \check{z}_i),$$

$$b \sum_{\omega} \beta(\omega)(1-\omega_i) \, d(\mathbf{z}, \omega) = b(1-z_i) \sum_{k=0}^{n} \beta_k (1-k/n) L(k, \quad n-1, \quad \check{z}_i).$$

Several particular cases of (7.2) hold special interest for us. If for some integer k_0 with $0 \leqq k_0 \leqq n$, (7.2) takes the form

$$\beta_{k_0} = 1, \qquad \beta_k = 0 \qquad\qquad k = 0, \cdots k_0 - 1, \; k_0 + 1, \cdots, \; n,$$

then the corresponding Bayes' solution (7.1) is a decision function of the form (6.10).

If for some constant p with $0 \leqq p \leqq 1$, expression (7.2) takes the form

$$\beta_k = \binom{n}{k} p^k (1-p)^{n-k},$$

then the corresponding Bayes' solution (7.1) is a decision function of the form (3.5) provided the Bayes' solution is required to be simple on the arbitrary part of its definition.

Referring back to Section 3, we have by the previous paragraph that each of the simple decision functions $\mathbf{t} = t_p$ for $0 \leqq p \leqq 1$, defined by (3.5), is a Bayes' solution for the compound decision problem. It can be shown (we omit the simple proof) that there always exists a value $0 < p' < 1$ and a determination of $t_{p'}$ for which the coefficient of p in (3.3) vanishes. Letting r be the constant value of $B(t_{p'}, p)$ it follows that $R(t_{p'}, \theta) \equiv r$ for every θ in Ω.

Since $t_{p'}$ has constant risk and is a Bayes' solution relative to a weight function which is positive for every θ in Ω, it follows that $t_{p'}$ is the admissible minimax decision function, unique in the sense of risk. That is

$$\sup_{\theta \text{ in } \Omega} R(\mathbf{t}, \theta) = \text{minimum for } \mathbf{t} = t_{p'},$$

and if $\mathbf{t}$ is any other decision function such that $\sup R(\mathbf{t}, \theta) = r$, then $R(\mathbf{t}, \theta) \equiv r$.

8. Admissibility. Since the minimax decison function $t_{p'}$ is simple and has constant risk r independent of n, it follows that

$$r = R(t_{p'}, \theta) \geqq \inf_{t} R(t, \theta) = \phi(\bar{\theta}) \qquad \text{for all } \theta \text{ in } \Omega,$$

and hence $r = \phi(p') = \max_p \phi(p)$.

If μ_0 and μ_1 are nondisjoint, $\phi(0) = \phi(1) = 0 < r$ and we conclude from Theorem 4 that, for any $0 < \epsilon < r$, the minimax decision function is ϵ-inadmissible (See [6] for definitions) for all sufficiently large n, since decision functions of the type $\mathbf{t}^*$ are ϵ-better.

The present paper has failed to exhibit a $\mathbf{t}^*$ which is admissible (or even to show the existence of such). This deficiency is remedied, at least asymptotically, by the fact that, for any $\epsilon > 0$ Theorems 4 and 5 together imply that any de-

cision function of the type $\mathbf{t}^*$ is ϵ-uniformly-best (relative to the class of R-invariant decision functions) for all sufficiently large n.

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THE EFFICIENCY OF TESTS[1]

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Summary. The efficiency of a family of tests is defined. Methods for evaluating the efficiency are discussed. The asymptotic efficiency is obtained for certain families of tests under assumptions which imply that the sample size is large.

1. Introduction. Let $w_n^{(1)}$ and $w_n^{(2)}$ be two tests each of which has the same fixed significance level for testing a hypothesis $\theta = \theta_1$, and which are defined for every sample size n. The relative efficiency of test $w_n^{(2)}$, or rather, of the sequence $\{w_n^{(2)}\}$, with respect to test $w_n^{(1)}$ has been defined as the ratio n_1/n_2, where n_2 is the least sample size required by a test of the sequence $\{w_n^{(2)}\}$ in order to achieve the same power for a given alternative $\theta = \theta_2$ as is achieved by the test in $\{w_n^{(1)}\}$ using a sample of size n_1. This is essentially the definition used by Pitman (see Noether [6], p. 241).

In Section 2 we extend this definition by replacing sequences of tests by arbitrary families of (nonsequential) tests and the parameters θ_1 and θ_2 by two arbitrary classes of distributions. The tests are regarded as general two-decision rules. If $N(\mathfrak{Z})$ is the least sample size used by a test in family $\mathfrak{Z}$ whose probabilities of the two kinds of error (corresponding to the two classes of distributions) do not exceed two given numbers, the ratio $N(\mathfrak{Z}_1)/N(\mathfrak{Z}_2)$ is defined as the relative efficiency of family $\mathfrak{Z}_2$ with respect to family $\mathfrak{Z}_1$.

In Section 3 it is pointed out that the determination of $N(\mathfrak{Z})$ is closely related to finding a test which maximizes the minimum power.

In Section 4 the problem of asymptotic efficiency is considered. In studies of the asymptotic efficiency of tests it is customary to consider a simple hypothesis $\theta = \theta_1$ (say) and a simple alternative, $\theta = \theta_2$, and to assume that as $n \to \infty$, θ_1 remains fixed and θ_2 approaches θ_1 in a certain way, for instance by setting $\theta_2 = \theta_1 + kn^{-1/2}$. Neither the restriction to simple hypotheses nor the assumption that θ_2 depends on n seems to be entirely adequate from the point of view of most applications.

In Section 4 of this paper a somewhat different approach is used. As a typical special case, consider a sequence of independent random variables with common cdf (cumulative distribution function) F. Let $\theta(F)$ be a real-valued function of F, and suppose that one or the other decision is undesirable according as $\theta(F) \leq \theta_1$ or $\theta(F) \geqq \theta_2$, where θ_1 and $\theta_2 > \theta_1$ are fixed numbers. Since θ_1 and θ_2 usually will be so chosen that neither decision is strongly preferred when $\theta_1 < \theta(F) < \theta_2$, small values of $\theta_2 - \theta_1$ are frequently of particular interest.

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Let $N(\mathfrak{z})$ denote the least sample size required by a test in family $\mathfrak{z}$ for which the probability of a wrong decision does not exceed α_1 if $\theta(F) \leqq \theta_1$, and does not exceed α_2 if $\theta(F) \geqq \theta_2$. Thus $N(\mathfrak{z})$ is a function of θ_1 and θ_2. Under suitable assumptions we derive asymptotic expressions for $N(\mathfrak{z})$ as $\delta = \theta_2 - \theta_1$ tends to zero, while α_1 and α_2 remain fixed. Illustrative examples are given in Section 5.

2. The efficiency of a family of tests. Let X be a random variable or a random vector with cdf F, which is assumed to belong to a class $\mathfrak{C}$ of cdfs. It is desired to decide, on the basis of n independent observations $\mathbf{x}_n = (x_1, \cdots, x_n)$ on X, between two alternative courses of action, A_1 and A_2. Let d_i denote the decision in favor of A_i, with $i = 1, 2$. Suppose there are given two disjoint subclasses $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of $\mathfrak{C}$ such that decision d_i is preferred if F is in $\mathfrak{C}_i$, for $i = 1, 2$. (In most applications one will choose the classes $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in such a way that neither decision is strongly preferred if F is neither in $\mathfrak{C}_1$ nor in $\mathfrak{C}_2$.)

Consider a decision rule, briefly referred to as a test, of the following type. Let E_n denote the space of points $\mathbf{x}_n$; if x_j is a vector with k components, E_n may be taken as the nk-dimensional Euclidean space. Let w_{1n} be a subset of E_n, and denote its complement by w_{2n}. (All sets of points $\mathbf{x}_n$ considered in this paper are assumed to be Borel sets, and all functions of $\mathbf{x}_n$ are understood to be Borel-measurable.) A test determined by the pair of sets $w_n = (w_{1n}, w_{2n})$ consists in taking n observations on X, and making decision d_i if the observed point $\mathbf{x}_n$ is in w_{in}, for $i = 1, 2$. This test will be referred to as the test w_n. A test w_n with $w_{in} \subset E_n$ will be called a test based on n observations.

Let α_1 and α_2 be two positive numbers. We shall say that the test w_n solves the problem $(\mathfrak{C}_1, \mathfrak{C}_2, \alpha_1, \alpha_2)$ if

$$P(\mathbf{X}_n \, \varepsilon \, w_{in} \mid F) \geqq 1 - \alpha_i \qquad \text{for all } F \text{ in } \mathfrak{C}_i, \qquad i = 1, 2, \tag{2.1}$$

where $\mathbf{X}_n = (X_1, \cdots, X_n)$ is a random vector with values in E_n, and $P(R \mid F)$ denotes the probability of relation R when $X_1, \cdots, X_n$ are independent with common cdf F.

Let $\mathfrak{z}$ be a family of tests. We shall mainly be concerned with families $\mathfrak{z}$ which, for every positive integer n, contain at least one test based on n observations. For example, $\mathfrak{z}$ may be the family of all tests w_n with

$$w_{1n} = \{\mathbf{x}_n : t_n(\mathbf{x}_n) < c\}, \qquad -\infty < c < \infty, \qquad n = 1, 2, \cdots,$$

where $\{t_n\}$ is a given sequence of functions, and $\{\mathbf{x}_n : R\}$ denotes the all set of points $\mathbf{x}_n$ such that relation R is satisfied.

Let $N(\mathfrak{z}) = N(\mathfrak{z}, \mathfrak{C}_1, \mathfrak{C}_2, \alpha_1, \alpha_2)$ be the least integer n such that the inequalities (2.1) are satisfied for some test in $\mathfrak{z}$. Thus $N(\mathfrak{z})$ is the least number of observations with which problem $(\mathfrak{C}_1, \mathfrak{C}_2, \alpha_1, \alpha_2)$ can be solved when we restrict ourselves to tests belonging to family $\mathfrak{z}$. If no test in $\mathfrak{z}$ satisfies (2.1), we set $N(\mathfrak{z}) = \infty$. The number $N(\mathfrak{z})$ will be termed the *efficiency index*, or simply the *index*, of family $\mathfrak{z}$ for problem $(\mathfrak{C}_1, \mathfrak{C}_2, \alpha_1, \alpha_2)$. Evidently $\mathfrak{z}$ may contain more than one test based on $N(\mathfrak{z})$ observations.

If $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are two families of tests and at least one of the indices $N(\mathfrak{Z}_i) = N(\mathfrak{Z}_i, \mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$, for $i = 1, 2$, is finite, the ratio

$$\text{eff}(\mathfrak{Z}_1/\mathfrak{Z}_2) = N(\mathfrak{Z}_2)/N(\mathfrak{Z}_1)$$

will be called the *efficiency* of family $\mathfrak{Z}_1$ relative to family $\mathfrak{Z}_2$ for problem $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$. The relative efficiency can thus equal any nonnegative rational number or infinity. This definition of relative efficiency of two tests is an extension of Pitman's definition of the same term (see Noether [6], p. 241).

Let $\mathfrak{Z}^*$ be the family of all tests $w_n = (w_{1n}, w_{2n})$, for all $n = 1, 2, \cdots$. If $N(\mathfrak{Z}^*) = \infty$, problem $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ has no solution (within the family $\mathfrak{Z}^*$). If $N(\mathfrak{Z}^*) < \infty$, and $\mathfrak{Z}$ is any subfamily of $\mathfrak{Z}^*$, then $\text{eff}(T/T^*)$ may be called the (absolute) efficiency of family $\mathfrak{Z}$ for problem $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$, provided we confine ourselves to tests in $\mathfrak{Z}^*$.

Clearly, $\text{eff}(\mathfrak{Z}/\mathfrak{Z}^*) \leqq 1$, the sign of equality holding if and only if $\mathfrak{Z}$ contains a test which solves problem $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ and is based on the least possible number of observations with which the problem can be solved by any test in $\mathfrak{Z}^*$.

All that has been said can immediately be extended to families of randomized tests. A randomized test is determined by a function $\phi_{1n}(\mathbf{x}_n)$, where $0 \leqq \phi_{1n}(\mathbf{x}_n) \leqq 1$. Let $\phi_{2n} = 1 - \phi_{1n}$. The test consists in taking n observations $\mathbf{x}_n$ and performing a random experiment whose two possible outcomes, e_1 and e_2, have probabilities $\phi_{1n}(\mathbf{x}_n)$ and $\phi_{2n}(\mathbf{x}_n)$, respectively. If event e_i occurs, decision d_i is made. A test determined by the pair of functions $\phi_n = (\phi_{1n}, \phi_{2n})$ will be referred to as the test ϕ_n.

If $\mathfrak{Z}$ is a family of randomized tests, the index $N(\mathfrak{Z}, \mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ is defined as the least n such that the relations

$$E(\phi_{in} \mid F) \geqq 1 - \alpha_i \qquad \text{for all } F \text{ in } \mathcal{C}_i, \qquad i = 1, 2,$$

are satisfied for some ϕ_n in $\mathfrak{Z}$. Here $E(\phi_{in} \mid F)$ denotes the expected value of $\phi_{in}(\mathbf{X}_n)$ when $X_1, \cdots, X_n$ are independent with the common cdf F. If randomized tests are admitted which do not use any observations, we could have $N(\mathfrak{Z}) = 0$. This trivial case can be excluded by assuming that $\alpha_1 + \alpha_2 < 1$.

The notion of efficiency could be extended to families of tests such that the choice of the number of observations also depends on a random experiment, the probabilities of whose outcomes may or may not depend on the observations. (In the former case we are dealing with sequential tests.) This paper is confined to families of nonsequential tests based on a nonrandom number of observations.

So far we have assumed, to simplify the exposition, that $X_1, X_2, \cdots$, is a sequence of independent, identically distributed random variables. Suppose, more generally, that for every n the random vector $\mathbf{X}_n = (X_1, \cdots, X_n)$ has a cdf G_n which belongs to a class $\mathcal{C}_n$. For every n there are given two disjoint subclasses, $\mathcal{C}_{1n}$ and $\mathcal{C}_{2n}$, of $\mathcal{C}_n$. We say that a test $\phi_n = (\phi_{1n}, \phi_{2n})$ solves problem $(\{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$ if

$$E(\phi_{in} \mid G_n) \geqq 1 - \alpha_i \qquad \text{for all } G_n \text{ in } \mathcal{C}_{in}, \qquad i = 1, 2,$$

where $E(\phi_{in} \mid G_n)$ denotes the expected value of $\phi_{in}(\mathbf{X}_n)$ when $\mathbf{X}_n$ has the cdf G_n. The definitions of $N(\mathfrak{Z}) = N(\mathfrak{Z}, \{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$ and eff $(\mathfrak{Z}_1/\mathfrak{Z}_2)$ are obvious extensions of the corresponding definitions in the special case.

3. The determination of $N(\mathfrak{Z})$. Let $\mathfrak{Z}$ be a given family of tests, randomized or not. Since a randomized test ϕ_n such that $\phi_n(\mathbf{x}_n) = 0$ or 1 for all $\mathbf{x}_n$ is equivalent to a nonrandomized test w_n, we may denote the tests in $\mathfrak{Z}$ by ϕ_n. Let $\mathfrak{Z}_n$ be the family of all tests in $\mathfrak{Z}$ which are based on n observations. Denote by $\mathfrak{Z}_{1n}$ the family of all tests ϕ_n in $\mathfrak{Z}_n$ for which

$$E(\phi_{1n} \mid G_n) \geqq 1 - \alpha_1 \qquad \text{for all } G_n \text{ in } \mathcal{C}_{1n}. \tag{3.1}$$

THEOREM 3.1. *Let*

$$M(\phi_n) = \sup_{G_n \varepsilon \mathcal{C}_{2n}} E(\phi_{1n} \mid G_n), \qquad M_n = \inf_{\phi_n \varepsilon \mathfrak{Z}_{1n}} M(\phi_n).$$

If $N(\mathfrak{Z}, \{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$ is finite, it is the least integer n for which $M_n \leqq \alpha_2$ and either $M_n < \alpha_2$, or $M_n = \alpha_2$ and $M(\phi_n) = \alpha_2$ for some ϕ_n in $\mathfrak{Z}_{1n}$.

The proof is left to the reader.

Due to the symmetry of the problem, the roles of $\mathcal{C}_{1n}$ and $\mathcal{C}_{2n}$ obviously can be interchanged.

Adapting a familiar terminology, we may say that a test which satisfies (3.1) has *level α_1 with respect to $\mathcal{C}_{1n}$*, and we may call $1 - M(\phi_n)$ the *minimum power of test ϕ_n with respect to $\mathcal{C}_{2n}$*. If there exists a test ϕ_n in $\mathfrak{Z}_{1n}$ such that $M(\phi_n) = M_n$, the test is said to *maximize the minimum power with respect to $\mathcal{C}_{2n}$*. Tests which maximize the minimum power can sometimes be obtained by a method due to Wald and explicitly applied to this problem by Lehmann ([3], Theorem 8.3). A proof of a special case of the theorem and illustrations of its use are given by Lehmann and Stein [4], [5]. Lehmann's theorem immediately applies to cases where $\mathfrak{Z}_n$ is the family of all tests ϕ_n, or of all tests ϕ_n which depend on a given function of $\mathbf{x}_n$. It can be extended to arbitrary families of tests.

4. Asymptotic efficiency. The rest of this paper considers the asymptotic behavior of the efficiency index $N(\mathfrak{Z})$ for certain families of tests in cases where the "distance" between the classes $\mathcal{C}_{1n}$ and $\mathcal{C}_{2n}$ is small. Let $\theta(G_n)$ be a real-valued function defined for all G_n in $\mathcal{C}_n$ and for every (or every sufficiently large) positive integer n. We assume that the set ω of values $\theta(G_n)$ when $G_n \varepsilon \mathcal{C}_n$ is an interval, finite or infinite, which is independent of n. Let θ_1 and θ_2 be two numbers in ω, with $\theta_1 < \theta_2$, and let $\mathcal{C}_{1n}$ and $\mathcal{C}_{2n}$ be the classes of all G_n in C_n such that $\theta(G_n) \leqq \theta_1$ and $\theta(G_n) \geqq \theta_2$, respectively.

Let $\{t_n(\mathbf{x}_n)\}$ for $n = 1, 2, \cdots$, be a given sequence of functions. Let $\mathfrak{Z}$ be the family of all tests $\phi_n^{(c)}$ with

$$\phi_{1n}^{(c)}(\mathbf{x}_n) = \begin{cases} 1 & \text{if } t_n(\mathbf{x}_n) < c; \\ \text{arbitrary} & \text{if } t_n(\mathbf{x}_n) = c; \\ 0 & \text{if } t_n(\mathbf{x}_n) > c; \end{cases} \qquad \begin{matrix} -\infty < c < \infty, \\ n = 1, 2, \cdots. \end{matrix} \tag{4.1}$$

For definiteness, we shall assume that $\phi_{1n}^{(c)} = 1$ if $t_n = c$, so that $\mathfrak{Z}$ is the family of nonrandomized tests $w_n^{(c)}$ with

$$w_{1n}^{(c)} = \{\mathbf{x}_n : \ t_n(\mathbf{x}_n) \leqq c\}, \qquad -\infty < c < \infty; \quad n = 1, 2, \cdots. \tag{4.2}$$

The asymptotic results to be derived for family (4.2) will also hold for any family (4.1).

Nothing will be changed if $\mathfrak{Z}$ consists of all tests (4.1) with n exceeding a fixed number n'. In this case the words "for every n" in the following assumptions should be read "for every $n > n'$."

We shall study the asymptotic behavior of $N(\mathfrak{Z}) = N(\mathfrak{Z}, \{\mathfrak{C}_{1n}\}, \{\mathfrak{C}_{2n}\}, \alpha_1, \alpha_2)$ when α_1, α_2 and θ_1 are held fixed and $\delta = \theta_2 - \theta_1$ tends to zero.

Assumption A. The functions $t_n(\mathbf{x}_n)$ are independent of δ.

Let

$$M_{1n}(x, \theta) = \inf_{\theta(G_n) \leqq \theta, G_n \in \mathfrak{C}_n} P(t_n \leqq x \mid G_n),$$

$$M_{2n}(x, \theta) = \sup_{\theta(G_n) \geqq \theta, G_n \in \mathfrak{C}_n} P(t_n \leqq x \mid G_n).$$

The functions $M_{in}(x, \theta)$ are nonincreasing in θ. Hence the conditions

$$P(\mathbf{X}_n \varepsilon w_{in} \mid G_n) \geqq 1 - \alpha_i \qquad \text{for all } G_n \varepsilon \mathfrak{C}_{in}, \qquad i = 1, 2,$$

are satisfied by test (4.2) if and only if

$$M_{1n}(c, \theta_1) \geqq 1 - \alpha_1, \qquad M_{2n}(c, \theta_1 + \delta) \leqq \alpha_2.$$

Assumption B. For every n, $\lim_{x \to \infty} M_{1n}(x, \theta_1) > 1 - \alpha_1$.

Let c_n be the least number which satisfies the inequalities

$$M_{1n}(c_n+, \theta_1) \geqq 1 - \alpha_1 \geqq M_{1n}(c_n-, \theta_1), \tag{4.3}$$

where c_n+ and c_n- are, respectively, the right and left limits. This number exists by Assumption B. In Theorem 3.1, we have $M_n = M_{2n}(c_n+, \theta_1 + \delta)$. Hence if $N = N(\mathfrak{Z})$ is finite and greater than 1,

$$M_{2N}(c_N+, \theta_1 + \delta) \leqq \alpha_2 \leqq M_{2,N-1}(c_{N-1}+, \theta_1 + \delta). \tag{4.4}$$

Assumption C. $\alpha_1 + \alpha_2 < 1$.

Assumption D. One of the following conditions is satisfied.

(a) For every n and every x, $M_{2n}(x, \theta)$ is continuous on the right in θ at $\theta = \theta_1$.

(b) For every n and every $\theta' \varepsilon \omega$ there exists a cdf $G_{\theta' n} \varepsilon \mathfrak{C}_n$ such that $\theta(G_{\theta' n}) = \theta'$, and for every x, $p_n(x, \theta) = P(t_n \leqq x \mid G_{\theta n})$ is continuous on the right in θ at $\theta = \theta_1$.

Lemma 4.1. *If Assumptions A, B, C, and D are satisfied, $N(\mathfrak{Z}) \to \infty$ as $\delta \to 0$.*

Proof. First suppose that part (a) of Assumption D is satisfied. Then for n fixed and $\delta \to 0$,

$$M_{2n}(c_n+, \theta_1 + \delta) \to M_{2n}(c_n+, \theta_1).$$

Since $M_{1n}(x, \theta) \leqq M_{2n}(x, \theta)$, we have, by (4.3) and Assumption C,

$$M_{2n}(c_n+, \theta_1) \geqq M_{1n}(c+, \theta_1) \geqq 1 - \alpha_1 > \alpha_2 .$$

Hence $N(\mathfrak{z})$ will be larger than any fixed n as $\delta \to 0$.

Now suppose that part (b) of Assumption D is satisfied. Then

$$M_{2n}(c_n+, \theta_1 + \delta) \geqq p_n(c_n, \theta_1 + \delta), \qquad p_n(c_n , \theta_1) \geqq M_{1n}(c_n+, \theta_1) \geqq 1 - \alpha_1 > \alpha_2 .$$

Since $p_n(c_n , \theta_1 + \delta) \to p_n(c_n , \theta_1)$ as $\delta \to 0$, we arrive at the same conclusion.

ASSUMPTION E. There exist a sequence $\{h_n(x)\}$ of nondecreasing, continuous functions, a positive number r, and functions $H_1(x)$ and $H_2(x, d)$, defined for every $d > 0$, such that

$$M_{1n}(h_n(x), \theta_1) \to H_1(x), \tag{4.5}$$

$$M_{2n}(h_n(x), \theta_1 + dn^{-r}) \to H_2(x, d) \tag{4.6}$$

as $n \to \infty$, for every x which is a point of continuity of $H_1(x)$ and $H_2(x, d)$, respectively, and every $d > 0$.

Note that $H_2(x, d)$ is necessarily nonincreasing in d, while $H_1(x)$ and $H_2(x, d)$ are nondecreasing in x.

ASSUMPTION F. (i) The equation $H_1(x) = 1 - \alpha_1$ has a unique root $x = a$, at which $H_1(x)$ is continuous.

(ii) The equation $H_2(a, d) = \alpha_2$ has a unique positive root $d = D(\alpha_1 , \alpha_2)$. The function $H_2(x, d)$ is continuous at $x = a$ for all d in a neighborhood of $D(\alpha_1 , \alpha_2)$.

Assumptions E and F have the following meaning. The function $M_{in}(x, \theta)$ is a bound of cdfs of t_n. The function $M_{in}(h_n(x), \theta)$ is the corresponding bound of the cdfs of $t'_n = f_n(t_n)$, where $f_n = h_n^{-1}$, the inverse of h_n. The function f_n is strictly increasing. The tests with w_{1n} determined by $t'_n \leqq c$ are equivalent to the tests (4.2). Thus Assumptions E and F require, roughly speaking, that a suitable function of t_n have a limiting distribution which satisfies certain regularity conditions.

THEOREM 4.1. *Let $\mathfrak{z}$ be a family of tests of the form* (4.1), *or the subfamily with $c = c'_n$ fixed in such a way that $h_n^{-1}(c'_n) \to a$ as $n \to \infty$. If Assumptions A through F are satisfied, then asymptotically as $\delta \to 0$,*

$$N(\mathfrak{z}) \sim \{\delta^{-1} D(\alpha_1 , \alpha_2)\}^{1/r}. \tag{4.7}$$

PROOF. We first assume that family $\mathfrak{z}$ is defined by (4.2). By the remark following Assumption F we may assume that $h_n(x) = x$. Then relations (4.5) and (4.6) are replaced by

$$M_{1n}(x, \theta_1) \to H_1(x), \tag{4.8}$$

$$M_{2n}(x, \theta_1 + dn^{-r}) \to H_2(x, d). \tag{4.9}$$

From (4.3), (4.8) and Assumption F(i) it follows that $c_n \to a$ as $n \to \infty$.

We now show that $N = N(\mathfrak{z})$ is finite for every $\delta > 0$, that is, $M_{2n}(c_n+, \theta_1 + \delta) \leqq \alpha_2$ for some n. By Assumption F we can choose $d > D =$

$D(\alpha_1, \alpha_2)$ so that $H_2(x, d)$ is continuous at $x = a$. For every $\delta > 0$ and n so large that $\delta > dn^{-r}$ we have

$$M_{2n}(c_n+, \theta_1 + \delta) \leqq M_{2n}(c_n+, \theta_1 + dn^{-r}).$$

By Assumptions E and F, the right side tends to $H_2(a, d)$ as $n \to \infty$, and $H_2(a, d) < \alpha_2$ by Assumption F. Hence N is finite.

We have to show that

$$\delta N^r \to D \quad \text{as} \quad \delta \to 0. \tag{4.10}$$

Suppose this is not true. Then there exists a sequence $\{\delta_k\}$ of positive numbers such that $\delta_k \to 0$ and $\delta_k N_k^r \to D' \neq D$ as $k \to \infty$, where $0 \leqq D' \leqq +\infty$ and N_k is the value of N for $\delta = \delta_k$. By (4.4) we have for every k

$$M_{2N_k}(c_{N_k}+, \theta_1 + \delta_k) \leqq \alpha_2 \leqq M_{2, N_k-1}(c_{N_k-1}+, \theta_1 + \delta_k). \tag{4.11}$$

First assume $D' < D$. By Assumption F(ii) there exists a number D'' such that $D' < D'' < D$ and $H_2(x, D'')$ is continuous at $x = a$. For k sufficiently large we have $\delta_k N_k^r < D''$ and hence

$$M_{2N_k}(c_{N_k}+, \theta_1 + \delta_k) \geqq M_{2N_k}(c_{N_k}+, \theta_1 + D''N_k^{-r}). \tag{4.12}$$

As $k \to \infty$, we have (by Lemma 4.1) $N_k \to \infty$ and $c_{N_k} \to a$. Hence

$$M_{2N_k}(c_{N_k}+, \theta_1 + D''N_k^{-r}) \to H_2(a, D'') > \alpha_2. \tag{4.13}$$

But relations (4.12) and (4.13) contradict (4.11). Hence $D' \geqq D$.

If we assume $D' > D$, a similar argument starting with the right member of (4.11) leads again to a contradiction. This completes the proof for family (4.2).

An inspection of the proof shows that (4.7) also holds for any family (4.1) and the subfamily with $c = c_n'$ fixed, as stated in the theorem.

We observe that the argument will not be essentially affected if the factor n^{-r} in (4.6) is replaced by an arbitrary decreasing function $k(n)$ which tends to zero as $n \to \infty$. Then $N(\mathfrak{z}) \sim k^{-1}(\delta/D)$.

Let $\{t_{1n}\}$ and $\{t_{2n}\}$ be two sequences of statistics, and denote by $\mathfrak{z}_1$ and $\mathfrak{z}_2$ the corresponding families of tests of the form (4.1). Suppose that Assumptions A through F are satisfied by both families. If r_i and $D_i(\alpha_1, \alpha_2)$, for $i = 1, 2$, denote the values of r and $D(\alpha_1, \alpha_2)$ for the two families, an application of Theorem 4.1 gives immediately

THEOREM 4.2. *Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be two families of tests of the form* (4.1) *which both satisfy Assumptions A through F. Then as $\delta \to 0$,*

$$\text{eff}\,(\mathfrak{z}_1/\mathfrak{z}_2) = \frac{N(\mathfrak{z}_2)}{N(\mathfrak{z}_1)} \sim \frac{D_2(\alpha_1, \alpha_2)^{1/r_2}}{D_1(\alpha_1, \alpha_2)^{1/r_1}}\, \delta^{(r_2 - r_1)/r_1 r_2}.$$

Thus if $r_1 < r_2$, the efficiency of family $\mathfrak{z}_1$ relative to family $\mathfrak{z}_2$ tends to zero. If $r_1 = r_2 = r$,

$$\lim_{\delta \to 0} \text{eff}\,(\mathfrak{z}_1/\mathfrak{z}_2) = (D_2(\alpha_1, \alpha_2) / D_1(\alpha_1, \alpha_2))^{1/r}.$$

Consider now the particular case where the distribution of t_n depends on G_n only through $\theta(G_n)$. We shall write $P(t_n \leqq x \mid \theta)$ for $P(t_n(\mathbf{X}_n) \leqq x \mid G_n)$ when $\theta(G_n) = \theta$. In addition we shall assume that the power $P(t_n > c \mid \theta)$ does not decrease as θ increases; a "reasonable" test can be expected to have this property.

ASSUMPTION B′. The distribution of $t_n(\mathbf{X}_n)$ depends on G_n only through $\theta(G_n)$, and $P(t_n \leqq x \mid \theta)$ is a nonincreasing function of θ.

If Assumption B′ is satisfied, we have $M_{1n}(x, \theta) = M_{2n}(x, \theta) = P(t_n \leqq x \mid \theta)$. Assumption B is then satisfied for every $\alpha_1 > 0$. Assumption D is implied by Assumption D′. For every x, $P(t_n \leqq x \mid \theta)$ is continuous on the right in θ at $\theta = \theta_1$.

In Assumptions E and F we have $H_1(x) = H_2(x, 0)$. If we let $H(x, d) = H_2(x, d)$, then $D(\alpha_1, \alpha_2)$ is the unique positive root of the equation $H^{-1}(\alpha_2, D) = H^{-1}(1 - \alpha_1, 0)$. Theorems 4.1 and 4.2 hold with Assumptions B and D replaced by B′ and D′.

In many applications the statistic t_n or a function of t_n will be asymptotically normally distributed. Let

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy. \tag{4.14}$$

ASSUMPTION E′. There exist a sequence $\{g_n(t)\}$ of everywhere increasing functions, a positive number r, and two functions $\mu(\theta)$ and $\sigma(\theta)$ defined for $\theta \in \omega$, such that

$$P\left\{ n^r \frac{g_n(t_n) - \mu(\theta_1 + dn^{-r})}{\sigma(\theta_1 + dn^{-r})} \leqq x \;\middle|\; \theta_1 + dn^{-r} \right\} = \Phi(x) \tag{4.15}$$

as $n \to \infty$, for every x and every $d \geqq 0$.

ASSUMPTION F′. The function $\mu(\theta)$ has at $\theta = \theta_1$ a right derivative $\mu'(\theta_1) > 0$. The function $\sigma(\theta)$ is positive and continuous on the right at $\theta = \theta_1$.

By Assumption F′ we can write

$$\mu(\theta_1 + dn^{-r}) = \mu(\theta_1) + dn^{-r}\mu'(\theta_1)(1 + \epsilon_n), \qquad \sigma(\theta_1 + dn^{-r}) = \sigma(\theta_1)(1 + \epsilon_n'),$$

where $\epsilon_n \to 0$ and $\epsilon_n' \to 0$ as $n \to \infty$. Hence

$$n^r \frac{g_n(t_n) - \mu(\theta_1 + dn^{-r})}{\sigma(\theta_1 + dn^{-r})} = n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)(1 + \epsilon_n')} - d\frac{\mu'(\theta_1)(1 + \epsilon_n)}{\sigma(\theta_1)(1 + \epsilon_n')}$$

has the same limiting distribution as

$$n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)} - d\frac{\mu'(\theta_1)}{\sigma(\theta_1)}.$$

Thus if we replace x in (4.15) by $x - d\mu'(\theta_1) / \sigma(\theta_1)$, we obtain

$$P\left\{ n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)} \leqq x \;\middle|\; \theta_1 + dn^{-r} \right\} \to \Phi\left(x - d\frac{\mu'(\theta_1)}{\sigma(\theta_1)} \right).$$

Assumption E is now satisfied with $H_1(x) = \Phi(x)$ and $H_2(x, d) = \Phi(x - d\mu'(\theta_1) / \sigma(\theta_1))$.

Let $\lambda(u)$ be defined by

$$u = 1 - \Phi(\lambda(u)) = \Phi(-\lambda(u)). \tag{4.16}$$

Then Assumption F is satisfied with

$$D(\alpha_1, \alpha_2) = \frac{\sigma(\theta_1)}{\mu'(\theta_1)} \{\lambda(\alpha_1) + \lambda(\alpha_2)\}.$$

Hence we can state

THEOREM 4.3. *Let $\mathfrak{Z}$ be a family of tests of the form* (4.1) *or the subfamily with $c = c_n'$ fixed in such a way that $n^r\sigma(\theta_1)^{-1}[g_n(c_n') - \mu(\theta_1)] \to \lambda(\alpha_1)$. If Assumptions A, B', C, D', E', F' are satisfied, then asymptotically as $\delta \to 0$,*

$$N(\mathfrak{Z}) \sim \left(\frac{\sigma(\theta_1)}{\mu'(\theta_1)}\right)^{1/r} \left(\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right)^{1/r}. \tag{4.17}$$

If $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are two families of tests which both satisfy the conditions of Theorem 4.3 with the same number r, we have

$$\lim_{\delta \to 0} \text{eff}\,(\mathfrak{Z}_1/\mathfrak{Z}_2) = \left(\frac{\mu_1'(\theta_1)}{\sigma_1(\theta_1)}\right)^{1/r} \Big/ \left(\frac{\mu_2'(\theta_1)}{\sigma_2(\theta_1)}\right)^{1/r}, \tag{4.18}$$

where $\mu_i'(\theta_1)$ and $\sigma_i(\theta_1)$ are the values of $\mu'(\theta_1)$ and $\sigma(\theta_1)$ for family $\mathfrak{Z}_i$.

Thus in this case the asymptotic relative efficiency is independent of α_1 and α_2.

Relation (4.18) is essentially due to Pitman, who obtained an analogous result (for $r = \frac{1}{2}$) under somewhat different assumptions. Pitman's result was extended by Noether [7].

Assumption A (t_n independent of δ) is somewhat restrictive. Thus if $\mathcal{C}_n$ is a class of distributions depending only on the parameter θ, the most powerful test for testing the hypothesis $\theta = \theta_1$ against the alternative $\theta = \theta_1 + \delta$ will in general depend on δ. If Assumption A is dropped, $M_{in}(x, \theta)$ and c_n will depend on δ. Theorem 4.1 can be extended to this case by suitably modifying the assumptions. We shall state a corresponding theorem for the special case where the distribution of t_n depends only on $\theta(G_n)$ (and on δ through t_n), and a function of t_n has a normal limiting distribution.

To emphasize the dependence on δ we shall write $t_{\delta n}$ for t_n and $c_n(\delta)$ for c_n. Let Assumption B' be satisfied.

ASSUMPTION D''. For every n,

$$P(t_{\delta n} \leqq x \mid \theta_1 + \delta) - P(t_{\delta n} \leqq x \mid \theta_1) \to 0 \tag{4.19}$$

as $\delta \to 0$, uniformly in x.

LEMMA 4.2. *If Assumptions B', C, and D'' are satisfied, $N(\mathfrak{Z}) \to \infty$ as $\delta \to 0$.*

PROOF. After substituting $c_n(\delta)$ for x in (4.19), the proof parallels that of Lemma 4.1.

ASSUMPTION E''. There exist functions $g_n(t, \delta)$, defined for $\delta > 0$ and $n = 1, 2, \cdots$, which are strictly increasing in t; two positive numbers r and δ_1; and two functions $\mu(\theta, \delta)$ and $\sigma(\theta, \delta)$ defined for $\theta \in \omega$ and $\delta > 0$, all such that

$$P\left\{n^r \frac{g_n(t_{\delta n}, \delta) - \mu(\theta, \delta)}{\sigma(\theta, \delta)} \leqq x \mid \theta\right\} \to \Phi(x)$$

as $n \to \infty$ for every x, uniformly for $\theta_1 \leqq \theta < \theta_1 + \delta_1$ and $0 < \delta < \delta_1$.

ASSUMPTION F″. The limit

$$\mu'(\theta_1, 0) = \lim_{0<\delta\to 0} \frac{\mu(\theta_1 + \delta, \delta) - \mu(\theta_1, \delta)}{\delta}$$

exists and is positive. Also, $\sigma(\theta, \delta)$ is positive and continuous on the right at $(\theta_1, 0)$, that is, $\sigma(\theta, \delta) \to \sigma(\theta_1, 0) > 0$ as $\theta \to \theta_1$ and $\delta \to 0$, with $\theta \geqq \theta_1$ and $\delta > 0$.

THEOREM 4.4. *Let $\mathfrak{Z}$ be a family of tests of the form (4.1) or the subfamily with $c = c_n(\delta)$. If Assumptions B′, C, D″, E″, F″ are satisfied, then asymptotically as $\delta \to 0$,*

$$N(\mathfrak{Z}) \sim \left(\frac{\sigma(\theta_1, 0)}{\mu'(\theta_1, 0)}\right)^{1/r} \left(\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right)^{1/r}.$$

PROOF. Assumption E″ implies

$$n^r \frac{g_n(c_n(\delta), \delta) - \mu(\theta_1, \delta)}{\sigma(\theta_1, \delta)} \to \lambda(\alpha_1)$$

as $n \to \infty$, uniformly for $0 < \delta < \delta_1$. The rest of the proof is similar to the proofs of Theorems 4.1 and 4.3.

5. Illustrations. Three examples are offered.

EXAMPLE 1. *A test for regression.* Let G_n be the cdf of n independent, normally distributed random variables with common variance σ^2 and means $EX_j = j\xi$ for $j = 1, \cdots, n$. Let $\theta = \theta(G_n) = \xi/\sigma$, with $\theta_1 = 0$ and $\theta_2 = \delta$. Let

$$t_n(\mathbf{x}_n) = \frac{\sum jx_j}{\sqrt{\sum i^2 \sum x_j^2 - (\sum jx_j)^2}} = \frac{y_n}{\sqrt{\sum x_j^2 - y_n^2}},$$

where $y_n = \sum jx_j / \sqrt{\sum j^2}$ and all summations are from 1 to n. Here y_n is normally distributed with mean $\xi\sqrt{\sum j^2}$ and variance σ^2. Also, $\sqrt{(\sum x_j^2 - y_n^2)/n}$ tends to σ in probability. Hence $\sqrt{n}t_n$ is asymptotically normal $(\theta\sqrt{\sum j^2}, 1)$. Observing that $\sum j^2 \sim n^3/3$, it is easy to verify that the conditions of Theorem 4.3 are satisfied with

$$g_n(t) = \sqrt{3}\, t/n, \qquad r = 3/2, \qquad \mu(\theta) = \theta, \qquad \sigma(\theta) = \sqrt{3}.$$

Thus

$$N \sim 3^{1/3} \left(\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right)^{2/3}.$$

EXAMPLE 2. *A one-sided test of fit.* Let $\theta(F) = \sup_{-\infty<x<\infty} \{F_0(x) - F(x)\}$, where $F_0(x)$ is a fixed continuous cdf. Let $F(x)$ belong to the class $\mathfrak{C}$ consisting of F_0 and all continuous cdfs F with $\theta(F) > 0$. Let $F_n(x)$ be the empirical cdf of a random sample of n observations from F, and let $t_n = \sqrt{n}\theta(F_n)$. For deciding

whether $F = F_0$ or $\theta(F) > 0$, consider a test which accepts the former alternative when $t_n < c$. Smirnov has shown (see, e.g., Feller [2]) that as $n \to \infty$,

$$P(t_n < x \mid F_0) \to 1 - e^{-2x^2}, \qquad x > 0.$$

Birnbaum [1] obtained the best upper and lower bounds for the power of the test in the class of continuous cdfs with $\theta(F)$ fixed. From Birnbaum's Theorem 1 we have for $0 < x < \delta\sqrt{n}$,

$$\sup_{\theta(F)=\delta} P(t_n < x \mid F) = \sup_{\delta \le v \le 1} Q_n(n^{-1/2}x, \delta, v),$$

where

$$Q_n(\epsilon, \delta, v) = \sum_{i=j+1}^{n} \binom{n}{i} (v - \delta)^i (1 - v + \delta)^{n-i}, \qquad j = [n(v - \epsilon)].$$

For $x \geqq \delta\sqrt{n}$, $\sup_{\theta(F)=\delta} P(t_n < x \mid F) = 1$.

The function $Q_n(\epsilon, \delta, v)$ is decreasing in δ. Hence if $\delta < \delta'$,

$$\sup_{\delta \le v \le 1} Q_n(\epsilon, \delta, v) \geqq \sup_{\delta' \le v \le 1} Q_n(\epsilon, \delta, v) \geqq \sup_{\delta' \le v \le 1} Q_n(\epsilon, \delta', v).$$

Thus for $0 < x < \delta\sqrt{n}$,

$$M_{2n}(x, \delta) = \sup_{\theta(F) \ge \delta} P(t_n < x \mid F) = \sup_{\delta \le v \le 1} Q_n(n^{-1/2}x, \delta, v).$$

For $x \leqq 0$, $M_{2n}(x, \delta) = 0$. For $x \geqq \delta\sqrt{n}$, $M_{2n}(x, \delta) = 1$.

If v is fixed, $0 < v < 1$, we have for $n \to \infty$,

$$Q_n(n^{-1/2}x, n^{-1/2}d, v) \to \Phi\left(\frac{x - d}{\sqrt{v(1 - v)}}\right), \qquad 0 < x < d.$$

Using this result it can be shown that as $n \to \infty$,

$$M_{2n}(x, n^{-1/2}d) \to \Phi(2x - 2d), \qquad 0 < x < d.$$

It can now be verified that the conditions of Theorem 4.1 are satisfied with $h_n(x) = x$ and

$$H_1(x) = \begin{cases} 0, & x \leqq 0, \\ 1 - e^{-2x^2}, & x > 0; \end{cases} \qquad H_2(x, d) = \begin{cases} 0, & x \leqq 0, \\ \Phi(2x - 2d), & 0 < x < d, \\ 1, & x \geqq d. \end{cases}$$

That Assumption D(b) is satisfied can be seen from Birnbaum's Theorem 1 and the fact that $Q_n(\epsilon, \theta, v)$ is continuous in θ. Thus if $\alpha_1 + \alpha_2 < 1$ and $\alpha_2 < \frac{1}{2}$,

$$N(3) \sim \frac{1}{\delta^2}\left\{\sqrt{\tfrac{1}{2}\log\frac{1}{\alpha_1}} + \tfrac{1}{2}\lambda(\alpha_2)\right\}^2.$$

EXAMPLE 3. *The double-exponential distribution and the sign test.* Let $\mathbf{x}_n$ have the probability density $2^{-n}\exp(-\sum_{j=1}^{n}|x_j - \theta|)$. The most powerful test for testing $\theta = 0$ against $\theta = \delta > 0$ accepts $\theta = 0$ for small values of $t_{3n} = \sum_{j=1}^{n} a(x_j, \delta)$, where $a(x, \delta) = (|x| - |x - \delta|)/\delta$. Since $a(x, \delta)$ is a non-

decreasing function of x, it is easily seen that $P(t_{\delta n} \leqq c \mid \theta)$ is a nonincreasing function of θ.

As $n \to \infty$, $t_{\delta n}$ is asymptotically normal $(n\mu(\theta, \delta), n\sigma^2(\theta, \delta))$, where

$$\mu(\theta, \delta) = \begin{cases} -1 + (e^{\theta} - e^{\theta-\delta})/\delta, & \theta \leqq 0, \\ (2\theta - \delta)/\delta + (e^{-\theta} - e^{\theta-\delta})/\delta, & 0 \leqq \theta \leqq \delta, \\ 1 + (e^{-\theta} - e^{-\theta+\delta})/\delta, & \delta \leqq \theta; \end{cases}$$

$$\sigma^2(\theta, \delta) = \mu_2(\theta, \delta) - \mu(\theta, \delta)^2;$$

$$\mu_2(\theta, \delta) = \begin{cases} 1 + \{(4 - 2\delta)e^{\theta} - (4 + 2\delta)e^{\theta-\delta}\}/\delta^2, & \theta \leqq 0, \\ \{8 + 4(\theta - \frac{1}{2}\delta)^2\}/\delta^2 - (4 + 2\delta)(e^{-\theta} + e^{\theta-\delta})/\delta^2, & 0 \leqq \theta \leqq \delta, \\ 1 + \{(4 - 2\delta)e^{-\theta+\delta} - (4 + 2\delta)e^{-\theta}\}/\delta^2, & \delta \leqq \theta. \end{cases}$$

It can be verified that the conditions of Theorem 4.4 are satisfied with $g_n(t, \delta) = t/n$, $r = \frac{1}{2}$, $\mu'(0, 0) = 1$, and $\sigma(0, 0) = 1$. That the uniformity condition in Assumption E″ is satisfied can be seen from the fact that $E\{\,|a(X, \delta) - \mu(\theta, \delta)|^3 \mid \theta\} \,/\, \sigma(\theta, \delta)^3$ is bounded (since $a(x, \delta)$ is bounded); this implies uniform convergence by Liapunov's or Berry's bounds for the remainder term in the central limit theorem.

Thus if $\mathfrak{Z}_\delta$ denotes the family of the most powerful tests based on $t_{\delta n}$, we have $N(\mathfrak{Z}_\delta) \sim \delta^{-2}[\lambda(\alpha_1) + \lambda(\alpha_2)]^2$.

As $\delta \to 0$, $t_{\delta n}$ tends to $t_{0n} = 2m - n$, where m is the number of positive values x_j, for $j = 1, \cdots, n$. The family $\mathfrak{Z}_0$ of the tests based on t_{0n} (sign tests) has asymptotically the same efficiency index (up to order δ^{-2}), so that the sign test has asymptotic efficiency 1 for the present problem. This is not surprising in the light of a recent result of Ruist [8], who showed that the sign test is most powerful for discriminating between two symmetrical continuous distributions which can be regarded as approximations of double-exponential distributions.

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ON A THEOREM OF PITMAN

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Summary. A theorem by Pitman on the asymptotic relative efficiency of two tests is extended and some of its properties are discussed.

1. Introduction. The idea of the relative efficiency of one estimate with respect to another estimate of the same parameter is well established. This cannot be said, however, of the corresponding concept for two tests of the same statistical hypothesis. This paper is concerned with a definition of the relative efficiency of two tests which seems to be due to Pitman (see, e.g., [1] p. 241) and has been used in several recent papers.

Definition. Given two tests of the same size of the same statistical hypothesis, the *relative efficiency* of the second test with respect to the first is given by the ratio n_1/n_2, where n_2 is the sample size of the second test required to achieve the same power for a given alternative as is achieved by the first test with respect to the same alternative when using a sample of size n_1.

In general the ratio n_1/n_2 will depend on the particular alternative chosen (as well as on n_1). However, in the asymptotic case, this somewhat undesirable fact can be avoided. It might be argued that restriction to the asymptotic case is even more undesirable in itself, but the unfortunate fact remains that for many test procedures in current use the asymptotic power function is the only one available.

Now, at least for consistent tests, the power with respect to a fixed alternative is practically 1 if the number of observations is sufficiently large. Therefore, the power no longer provides a worthwhile criterion for preferring one test over another. On the other hand, it is possible to define sequences of alternatives changing with n in such a way that as $n \to \infty$ the power of the corresponding sequence of tests converges to some number less than 1. It seems then reasonable to define the asymptotic relative efficiency of the second test procedure with respect to the first test procedure as the limit of the corresponding ratios n_1/n_2.

A theorem due to Pitman allows us to compute this limit if certain general conditions are satisfied [1]. The purpose of this paper is to give an extension of Pitman's theorem and to discuss some of its properties. The derivation of the present version of the theorem follows Pitman's original method of proof. Since this proof has not appeared in print, full details are given.

2. Asymptotic power. Assume that we want to test the null hypothesis H_0: $\theta = \theta_0$ against alternatives H: $\theta > \theta_0$. As mentioned in the Introduction, we shall assume actually that a particular alternative $\theta = \theta_n$ changes with the sample size n in such a way that $\lim_{n\to\infty} \theta_n = \theta_0$.

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To be more definite, let the test be based on the static $T_n = T(x_1, \cdots, x_n)$. Let[1] $ET_n = \psi_n(\theta)$ and var $T_n = \sigma_n^2(\theta)$. Assume that

A. $$\psi_n'(\theta_0) = \cdots = \psi_n^{(m-1)}(\theta_0) = 0, \qquad \psi_n^{(m)}(\theta_0) > 0,$$

B. $$\lim_{n\to\infty} n^{-m\delta}\psi_n^{(m)}(\theta_0) / \sigma_n(\theta_0) = c > 0 \quad \text{for some } \delta > 0.$$

The indicated derivatives are assumed to exist. We shall consider the power of the test based on T_n with respect to the alternative H_1: $\theta_n = \theta_0 + k/n^\delta$, where k is an arbitrary positive constant. In addition to A and B we shall assume

C. $$\lim_{n\to\infty} \psi_n^{(m)}(\theta_n) / \psi_n^{(m)}(\theta_0) = 1, \qquad \lim_{n\to\infty} \sigma_n(\theta_n) / \sigma_n(\theta_0) = 1,$$

D. the distribution of $[T_n - \psi_n(\theta)] / \sigma_n(\theta)$ tends to the normal distribution with mean 0 and variance 1, uniformly in θ, with $\theta_0 \leqq \theta \leqq \theta_0 + d$ for some $d > 0$.

Let $\phi(\lambda) = \int_\lambda^\infty \exp(-\frac{1}{2}x^2)\,dx / \sqrt{2\pi}$ and find λ_α such that $\phi(\lambda_\alpha) = \alpha$. For sufficiently large n, a critical region of approximate size α is given by

$$T_n \geqq T_n(\alpha), \qquad [T_n(\alpha) - \psi_n(\theta_0)] / \sigma_n(\theta_0) = \lambda_\alpha .$$

The power of this test with respect to the alternative H_1 is given by

$$L_n(\theta_n) = P\{T_n \geqq T_n(\alpha) \mid \theta_n\} \sim \phi(t_n),$$

where $t_n = [\sigma_n(\theta_0)\lambda_\alpha + \psi_n(\theta_0) - \psi_n(\theta_n)] / \sigma_n(\theta_n)$. Now

$$\psi_n(\theta_n) = \psi_n(\theta_0 + k/n^\delta) = \psi_n(\theta_0) + (1/m!)(k/n^\delta)^m \psi_n^{(m)}(\hat{\theta}), \qquad \theta_0 < \hat{\theta} < \theta_n$$

and

$$t_n = \frac{\sigma_n(\theta_0)\lambda_\alpha - (1/m!)(k/n^\delta)^m[\psi_n^{(m)}(\hat{\theta})/\psi_n^{(m)}(\theta_0)]\psi_n^{(m)}(\theta_0)}{[\sigma_n(\theta_n)/\sigma_n(\theta_0)]\sigma_n(\theta_0)} \xrightarrow[n\to\infty]{} \lambda_\alpha - \frac{k^m c}{m!}.$$

Thus asymptotically, $L_n(\theta_n) \sim \phi(\lambda_\alpha - k^m c/m!)$.

It follows from the proof that condition (D) can be replaced by the somewhat weaker condition

D′. the distribution of $[T_n - \psi_n(\theta_n)] / \sigma(\theta_n)$ tends to the normal distribution with mean 0 and variance 1, both under the alternative hypothesis H_1 and under the null hypothesis $\theta_n = \theta_0$.

It is also clear that alternatives of the type $\theta_n < \theta_0$ or $|\theta_n| \neq \theta_0$, or the case when $\psi_n^{(m)}(\theta_0) < 0$, can be handled correspondingly.

Let $\gamma > \delta$ and consider the alternative H_2: $\theta_n = \theta_0 + k/n^\gamma$ an alternative which converges to θ_0 faster than H_1. Now

$$t_n \sim \lambda_\alpha - \lim_{n\to\infty} \frac{k^m}{m!n^{\gamma m}} \frac{\psi_n^{(m)}(\theta_0)}{\sigma_n(\theta_0)} = \lambda_\alpha - \lim_{n\to\infty} \frac{k^m}{m!n^{(\gamma-\delta)m}} \frac{\psi_n^{(m)}(\theta_0)}{n^{\delta m}\sigma_n(\theta_0)} = \lambda_\alpha ,$$

[1] J. Putter [4] has pointed out that the functions $\psi_n(\theta)$ and $\sigma_n^2(\theta)$ need not necessarily be the mean and variance of T_n, respectively, as long as conditions A, B, C, and D or D′ are satisfied.

and the power of our test with respect to this alternative H_2 is equal to the size of the critical region. The test cannot distinguish between H_0 and H_2. Similarly, if $\gamma < \delta$, the power of our test converges to 1.

3. Asymptotic relative efficiency of two tests. Assume now that we have two tests based on the statistics T_{1n} and T_{2n}. Assume further that $\delta_1 > \delta_2$ and consider the alternative $H_{1'}: \theta_n = \theta_0 + k/n^{\delta_1}$. It follows from our previous results that the second test is ineffective with respect to this alternative, while the power of the first test can be made as large as we please by choosing k sufficiently large. Therefore, the asymptotic relative efficiency of the second test with respect to the first test is zero.

Thus, from now on, we assume that $\delta_1 = \delta_2 = \delta$.

According to our definition of relative efficiency, the two tests must have identical power with respect to identical alternatives. The two tests have identical power if

$$k_1^{m_1} c_1/m_1! = k_2^{m_2} c_2/m_2! \tag{1}$$

The alternatives are identical if

$$k_1/n_1^{\delta} = k_2/n_2^{\delta}. \tag{2}$$

If now $m_1 = m_2 = m$, as it must be if $m_i\delta = \frac{1}{2}$ for $i = 1$ and 2, which is true in most cases, we can proceed as follows. From (1) and (2) we have

$$\frac{n_1}{n_2} = \left(\frac{k_1}{k_2}\right)^{1/\delta} = \left(\frac{c_2}{c_1}\right)^{1/m\delta} = \lim_{n\to\infty} \frac{(1/n)[\psi_{2n}^{(m)}(\theta_0) / \sigma_{2n}(\theta_0)]^{1/m\delta}}{(1/n)[\psi_{1n}^{(m)}(\theta_0) / \sigma_{1n}(\theta_0)]^{1/m\delta}} = \lim_{n\to\infty} \frac{R_{2n}^{1/m\delta}(\theta_0)}{R_{1n}^{1/m\delta}(\theta_0)} = E_{21}$$

where

$$R_{in}(\theta) = \psi_{in}^{(m)}(\theta) / \sigma_{in}(\theta), \qquad i = 1, 2. \tag{3}$$

Pitman has called the quantity $R_{in}^{1/m\delta}(\theta_0)$ the *efficacy* of the ith test in testing the hypothesis $H_0: \theta = \theta_0$. Thus we get

PITMAN'S THEOREM. *The asymptotic relative efficiency of two tests satisfying* A, B, C, *and* D *or* D′, *with* $\delta_1 = \delta_2$ *and* $m_1 = m_2$, *is given by the limit of the ratio of the efficacies of the two tests.*

For $m = 1$ and $\delta = \frac{1}{2}$, this theorem reduces to the one quoted in [1]. *If* $m\delta = \frac{1}{2}$,

$$E_{21} = \lim_{n\to\infty} R_{2n}^2(\theta_0) / R_{1n}^2(\theta_0). \tag{4}$$

If, in addition,

$$\lim_{n\to\infty} \psi_{2n}^{(m)}(\theta_0) / \psi_{1n}^{(m)}(\theta_0) = 1, \tag{5}$$

then (4) reduces to $E_{21} = \lim_{n\to\infty} \sigma_{1n}^2/\sigma_{2n}^2$.

This is the usual expression for measuring the asymptotic relative efficiency of two estimates of the same parameter. Thus, only if (5) is satisfied can we use the ratio of the variances of the two test statistics as a measure of the asymptotic relative efficiency of the two tests. In particular, if T_{1n} and T_{2n} are unbiased

estimates of the parameter θ, (5) is satisfied with $m = 1$, and E_{21} is the same as the asymptotic relative efficiency of T_2 and T_1 used as estimators of θ.

4. Comparison with another definition of relative efficiency. Another definition of the relative efficiency of two tests in current use (see, e.g., [2] p. 597) is based on the ratio of the respective sample sizes under the assumption that the power functions of the two tests have equal slope at $\theta = \theta_0$.

We shall show that if $m = 1$ and $\delta = \frac{1}{2}$, the two definitions give the same value for the asymptotic relative efficiency, provided a very general condition is satisfied.

Under the conditions of Pitman's theorem we have, as before, $L_n(\theta) \sim \phi(t_n)$ where

$$t_n = [\lambda_\alpha \sigma_n(\theta_0) + \psi_n(\theta_0) - \psi_n(\theta)] / \sigma_n(\theta).$$

If $L_n'(\theta)$ converges uniformly to some limit, this limit must be $d\phi(t_n)/d\theta$. Actually, the exact form of $L_n'(\theta)$ will rarely be known, so that the uniform convergence cannot be investigated. However, even in this case it is customary to replace $L_n'(\theta)$ by $d\phi(t_n)/d\theta$ in computing the ratio of the slopes of the two power functions. Thus it seems reasonable to compare the asymptotic relative efficiency based on the slopes $d\phi(t_n)/d\theta$ with E_{21}. Now

$$\left.\frac{d\phi(t_n)}{d\theta}\right|_{\theta=\theta_0} = \frac{-1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}t_n^2\right) \left.\frac{\partial t_n}{\partial \theta}\right|_{\theta=\theta_0}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\lambda_\alpha^2\right) \left[\frac{\psi_n'(\theta_0)}{\sigma_n(\theta_0)} + \lambda_\alpha \frac{\sigma_n'(\theta_n)}{\sigma_n(\theta_0)}\right] \sim \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\lambda_\alpha^2\right) cn^{1/2},$$

provided $\sigma_n'(\theta_0)/\sigma_n(\theta_0) = o(\sqrt{n})$, which is very generally true. The requirement that the two power functions have equal slope at the point $\theta = \theta_0$ becomes $c_1\sqrt{n_1} = c_2\sqrt{n_2}$, so that the asymptotic relative efficiency according to this definition is again given by

$$(n_1/n_2) = (c_2/c_1)^2 = E_{21}.$$

5. Efficacy of a test. Still assuming that $m = 1$ and $\delta = \frac{1}{2}$, it is interesting to investigate more closely the efficacy R_n^2, where R_n is given by (3). Consider the function $t = \psi_n(u)$ determined by $E\, T_n = \psi_n(\theta)$. Unless $\psi_n(\theta) = \theta$, T_n is not an unbiased estimate of θ, but may be considered an unbiased estimate of the ficticious parameter $\tau = \psi_n(\theta)$.

Let $u = \varphi_n(t)$ be the inverse of $t = \psi_n(u)$ and define the statistic $U_n = \varphi_n(T_n)$. Then we may write

$$U_n - \theta = \varphi_n(T_n) - \varphi_n(\tau) = (T_n - \tau)\, \varphi_n'(\tau) + \cdots.$$

If it is permissible to neglect terms of higher order in $T_n - \tau$, we find $E\, U_n \sim \theta$ and

$$\operatorname{var} U_n \sim [\varphi_n'(\tau)]^2 \sigma_n^2(\theta) = \sigma_n^2(\theta) / [\psi_n'(\theta)]^2 = R_n^{-2}(\theta).$$

Thus, if the above conditions are satisfied, asymptotically the efficacy of T_n is the reciprocal of the variance of an asymptotically unbiased estimate[2] of θ based on T_n.

Now, under the regularity conditions for the Cramér-Rao inequality, var $U_n \geqq 1/nE(\partial \log f/\partial\theta)^2$. Therefore

$$R_n^2(\theta) \leqq nE(\partial \log f/\partial\theta)^2.$$

Thus we may use the quantity $R_n^2(\theta) \,/\, nE(\partial \log f/\partial\theta)^2$ as a measure of the asymptotic efficiency of the test based on the statistic T_n.

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[2] Essentially this same result has also been obtained by Stuart [3]. However Stuart uses it even in some cases for which $m = 1$ but $\delta \neq \frac{1}{2}$. That it is then no longer correct can be seen easily from the generalized form of Pitman's Theorem given in Section 3 of this paper.

AN EXTENSION OF WALD'S THEORY OF STATISTICAL DECISION FUNCTIONS[1]

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1. Introduction. The material of the present paper was developed, during the spring of 1953, primarily to meet pedagogical needs. It is similar to the contents of Chapters 2 and 3 of Wald's book [1]. The results are an extension of Wald's theory in the sense that some requirements of boundedness or even finiteness of the loss function are removed. Moreover, Wald's requirements of equicontinuity are replaced by a requirement of lower semicontinuity of the loss function.

In the first part of the paper it is shown that, under suitable assumptions, the set $\mathfrak{D}$ of all decision functions can be identified with a convex subset of a certain topological vector space. If further assumptions are made on the loss function, the risk functions become lower semicontinuous linear functions defined on $\mathfrak{D}$. It is then easy to give conditions under which $\mathfrak{D}$, or some subset D of $\mathfrak{D}$, is compact.

The next section is devoted to proofs that convexity and compactness of the space of decision functions, together with lower semicontinuity of the risk functions, imply completeness of the intersection of the class of Bayes' solutions in the wide sense with the closure of the class of Bayes' solutions.

The methods of proof differ very little from the methods used by Wald [1], though it has been necessary to use slightly more general topological methods, for instance, to prove compactness instead of sequential compactness. Although it might be possible to extend the proofs given by Wald [1] or Karlin [2], [3] to the case considered here, it is on the whole simpler and shorter to start from the basic elementary lemmas.

2. Assumptions on the decision problem. In this section weakened forms of Assumptions 3.1 to 3.6 of [1] are stated. Let $X = \{X_i\}$ for $i = 1, 2, \cdots$ be a set of random elements, not necessarily real or even vector valued. Let $\mathfrak{X}$ be the space of values of X and let Ω be an arbitrary set of indices. We will suppose that there is given on $\mathfrak{X}$ a σ-field $\mathfrak{A}$ with respect to which all the X_i's are measurable, and that to each $\omega \in \Omega$ corresponds a probability distribution on $\mathfrak{A}$.

If the variables $X_{i_1}, X_{i_2}, \cdots, X_{i_k}$ are observed in this order, we will say that $\lambda = \{i_1, i_2, \cdots, i_k\}$ is observed and restrict the notation λ to ordered sets of indices which can be observed in the order given by λ, the first variable observed being X_{i_1}, the second X_{i_2}, and so on. The variables $\{X_{i_1}, \cdots, X_{i_k}\}$ determine on $\mathfrak{X}$ a smallest σ-field $\mathfrak{A}_\lambda \subset \mathfrak{A}$ with respect to which they are measurable. For all practical purposes it is equivalent to say that an $\mathfrak{A}$-measurable function $f(x)$ is $\mathfrak{A}_\lambda$-measurable or that it is a function of $\{X_{i_1}, \cdots, X_{i_k}\}$ only.

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At a certain stage λ of experimentation (including the start), the statistician has to choose an element among a set of available actions. This set Δ_λ includes the available terminal decisions T_λ as well as the decisions on how to continue experimentation, say J_λ. The set J_λ will be identified with the set of indices of variables which could be observed in the next step. We assume the following.

ASSUMPTION 1. Whatever may be λ, the set J_λ is finite.

This condition implies that the set of variables which can be observed in not more than n steps is finite. Let Λ_n be the corresponding set of λ's and let $\Lambda = \bigcup_n \Lambda_n$. Let $\mathfrak{A}_n$ be the smallest σ-field containing all the $\mathfrak{A}_\lambda$ for $\lambda \in \Lambda_n$. To simplify further formulas, we denote by P_ω^n and P_ω^λ the contractions of P_ω to $\mathfrak{A}_n$ and $\mathfrak{A}_\lambda$, respectively.

ASSUMPTION 2. For every n there exists on $\mathfrak{A}_n$ a σ-finite measure μ_n such that, whatever may be $\omega \in \Omega$, the probability measure P_ω^n is absolutely continuous with respect to μ_n.

Assumption 2 will be the only restriction placed upon the strategies of nature. It is implied by (3.1) or (3.2) of [1], but is strictly weaker than either (3.1) or (3.2). Discussion of Assumption 2 is deferred until after the statement of Assumption 6.

ASSUMPTION 3. For every $\lambda \in \Lambda$, the terminal space T_λ is a metrisable space, locally compact and the union of a denumerable family of compact spaces.

The space Δ_λ will be considered as topological sum of T_λ and J_λ, this last set being supplied with a discrete topology.

If δ is a certain decision function, for a given n, a given $\lambda \in \Lambda_n$, and an $x \in \mathfrak{X}$, we can consider the probability that λ be observed and that the next decision belongs to a specified subset S of Δ_λ. This defines a measure denoted below by $\varphi(\lambda, x, \delta)$ on a σ-field of subsets of Δ_λ. It will always be assumed that this σ-field is the σ-field of Borel subsets of Δ_λ. If u is a numerical function defined on Δ_λ, its integral with respect to the measure $\varphi(\lambda, x, \delta)$ will be denoted by $\varphi(\lambda, x, \delta) \circ u$.

Thus if u is the indicator of a Borel set $\bar{D}^t \subset T_\lambda$, then $\varphi(\lambda, x, \delta) \circ u$ corresponds exactly to the $p(d_1^e, d_2^e, \cdots, d_k^e, \bar{D}^t \mid x, \delta)$ defined in Formula (1.3) of [1]. In the present notation, λ replaces the ordered set of d_i^e and u replaces $\bar{D}^t$. It will be convenient to denote by a_λ the function identically equal to unity on T_λ and zero on J_λ, and by b_λ the function identically equal to unity on $\Delta_\lambda = T_\lambda \cup J_\lambda$. Furthermore, let K_λ denote the linear space of bounded continuous functions defined on Δ_λ and vanishing outside a compact subset of Δ_λ.

ASSUMPTION 4. Whatever $\lambda \in \Lambda$ and whatever $u \in K_\lambda$, the integral $\varphi(\lambda, x, \delta) \circ u$ is an $\mathfrak{A}_\lambda$-measurable function of x.

The purpose of this assumption is to give a meaning to integrals used below in the definition of a risk function. It also expresses the fact that $\varphi(\lambda, x, \delta)$ does not depend on variables which have not yet been observed.

ASSUMPTION 5. The cost of observing λ is a nonnegative, $\mathfrak{A}_\lambda$-measurable function $C(\lambda, x, \omega)$. If λ' is an initial segment of λ then $C(\lambda', x, \omega) \leqq C(\lambda, x, \omega)$. Moreover there exists a sequence $C_n^*(\omega)$ of nonnegative functions of ω such that $\lim_{n\to\infty} C_n^*(\omega) = \infty$ and $C_n^*(\omega) \leqq C(\lambda, x, \omega)$ for every λ having at least n elements.

ASSUMPTION 6. If the statistician does not reach a terminal decision in a finite

number of steps, he pays an infinite amount. If the statistician reaches a decision t in n steps by observing λ he pays a total amount

$$C(\lambda, x, \omega) + W(\omega, t; \lambda)h(\lambda, x, \omega),$$

where C is the cost function defined above and W and h are extended numerical functions satisfying the following conditions:

(i) $\inf_{\lambda \in \Lambda, t \in T_\lambda} W(\omega, t; \lambda) > -\infty$;

(ii) h is a nonnegative $\mathfrak{A}_\lambda$-measurable function such that $\sup_x h(\lambda, x, \omega) \leqq 1$ for every (λ, ω).

(iii) For each (λ, ω), the function W is a lower semicontinuous function of t on T_λ.

REMARK. It is possible to make W depend directly on x by introducing conditions of strong measurability. For instance $W(\omega, t, \lambda)h(\lambda, x, \omega)$ could be replaced by $\sum_{j=1}^{\infty} W_j(\omega, t, \lambda)h_j(\lambda, x, \omega)$, with W_j and h_j nonnegative.

Before entering further developments we will discuss Assumption 2 and introduce some simplifying notation. We notice first that in Assumption 2, μ_n could as well be taken finite instead of σ-finite. Since μ_n will not enter by itself in the next section, this is irrelevant. The properties actually used in the proofs are much weaker, and Assumption 2 could be relaxed as indicated below. Assume that μ_n has been chosen finite. For each $\lambda \in \Lambda_n$ let $\mathfrak{L}_\lambda$ be the space of equivalence classes of numerical functions defined on $\mathfrak{X}$, measurable with respect to $\mathfrak{A}_\lambda$ and μ_n integrable. If the norm of $f \in \mathfrak{L}_\lambda$ is defined by $\|f\| = \int |f| \, d\mu_n$, then $\mathfrak{L}_\lambda$ is a Banach space. The adjoint of $\mathfrak{L}_\lambda$ is, according to (2), identical with the space of μ_n-equivalence classes of bounded $\mathfrak{A}_\lambda$-measurable functions on $\mathfrak{X}$.

It is convenient to reduce the arbitrariness of μ_n and $\mathfrak{L}_\lambda$, and take instead of $\mathfrak{L}_\lambda$ the smallest L-space, say L_λ, containing the family $\{P_\omega^\lambda\}$. For the definition of this space and the properties used below, see [4] and [5]. Two $\mathfrak{A}_\lambda$-measurable functions φ_1 and φ_2 can be called P-equivalent if $\int |\varphi_1 - \varphi_2| \, dP_{\omega}^{\lambda} = 0$ for every $\omega \in \Omega$. Assumption 2 could then be replaced by the considerably weaker assumption that, whatever may be $\lambda \in \Lambda$, the space L_λ has for adjoint the space M_λ of P-equivalence classes of bounded $\mathfrak{A}_\lambda$-measurable functions. Thus such nonconventional families of distributions as the family of all discrete distributions on the interval [0, 1] could conceivably be introduced in the theory.

Even under its restricted form, Assumption 2 is more general than, for instance, (3.1) of [1] since it does not place such a strong restriction on the dependence between successive observations. This allows the consideration of different groups of random variables as vector variables which might be very strongly related. Since the cost of observing a vector might very well be different from the total cost of observing the components separately, and since moreover some of the spaces T_λ might be empty, the scheme considered here is at least as general as the scheme considered in [1].

The spaces L_λ and M_λ just defined will be used in the rest of the paper. Let L_λ^+ denote the positive cone of L_λ, and let $L_\lambda(a)$ be the sphere $L_\lambda(a) =$

$\{f: \|f\| \leqq a\}$. Furthermore, let $L_\lambda^+(a)$ be the common part of L_λ^+ and $L_\lambda(a)$. Similar notations will be used for the spaces M_λ and K_λ previously mentioned. If $s \in M_\lambda$ and $f \in L_\lambda$, let $f \circ s$ denote the integral $f \circ s = \int s(x)\, df$. Let $\mathfrak{M}_\lambda$ be the space of P-equivalence classes of $\mathfrak{A}_\lambda$-measurable finite or infinite numerical functions on $\mathfrak{X}$.

Let S be a subset of $\mathfrak{M}_\lambda$ such that, if $s_1, s_2 \in S$, then there exists $s_3 \in S$ satisfying $s_3 \geqq s_1$ and $s_3 \geqq s_2$. Assume moreover that the elements of S are all larger than a given $s_0 \in M_\lambda$. The assumption that M_λ is the adjoint of L_λ is known to be equivalent to the following: for each S with the precited properties, there exists an $\bar{s} \in \mathfrak{M}_\lambda$ such that $f \in L_\lambda^+$ implies $f \circ \bar{s} = \sup_{s \in S} f \circ s$. If $S \in M_\lambda(a)$, then $\bar{s} \in M_\lambda(a)$ and is uniquely determined. This $\bar{s}$ will be called the *supremum* of S and denoted by $\bar{s} = \sup_{s \in S} s = \sup S$. By convention if λ is the empty set we will take M_λ and L_λ to be the real line.

It will be necessary to distinguish between the equivalence classes belonging to M_λ and functions belonging to these equivalence classes. Thus, if $s \in M$, a function of the equivalence class of s will be denoted by $\bar{s}$ or by $s(x)$. Similar notation will be used for M_λ and $\widetilde{M}_\lambda$.

3. Representation of decision functions by families of linear mappings. If Assumptions 1 to 6 are satisfied, as far as the value of the risk is concerned a decision function is adequately described if the measures $\varphi(\lambda, x, \delta)$ are given up to a P-equivalence. It will therefore be convenient to identify decision functions which differ only on P-null sets. For a given $\lambda \in \Lambda$ the measures $\varphi(\lambda, x, \delta)$ define linear mappings from K_λ to M_λ. These linear mappings satisfy the following conditions (the subscript λ has been omitted for easier reading).

(a) $\varphi \circ u_1 + \varphi \circ u_2 = \varphi \circ (u_1 + u_2)$,

(b) $\varphi \circ \alpha u = \alpha \varphi \circ u$ for every real α,

(c) $\|\varphi \circ u\| \leqq \|u\|$,

(d) if $u \in K^+$, then $\varphi \circ u \in M^+$.

For any element $s \in M$, let $|s| = \sup(0, s) - \inf(0, s)$ and for any mapping satisfying properties (a, b, c) let $\nu_\varphi = \sup_{u \in K_\lambda^+(1)} |\varphi \circ u|$. It is easily seen that for a φ satisfying (a), (b), (c), and (d), we have

$$\nu_{\varphi_\lambda} = \varphi_\lambda \circ b_\lambda = \sup_{u \in K_\lambda^+(1)} \varphi_\lambda \circ u.$$

In these formulas the supremum is taken with the meaning as previously defined.

Let Δ_0 be the space of decisions available before experimentation starts. Let $\lambda = \{i_1, \cdots, i_k\}$ and $(\lambda, j) = \{i_1, \cdots, i_k, j\}$ with $j \in J_\lambda$, and let u_j be the indicator of j in Δ_λ. The decision procedures also satisfy:

(e) $\varphi_0 \circ \Delta_0 = 1$,

(f) $\varphi_\lambda \circ u_j = \varphi_{(\lambda, j)} \circ b_{(\lambda, j)}$,

(g) $\varphi(\lambda, x, \delta) \circ u = [\varphi(\lambda, x, \delta) \circ b_\lambda] \int_{\Delta_\lambda} u(t)\, dF_x(t)$,

where F_x is for each x a probability distribution on Δ_λ. It is not difficult to see that every system of measures satisfying conditions (a) to (g) defines an essentially unique decision procedure. The only condition not directly expressible in terms of equivalence classes is condition (g). The purpose of this section is to show that under reasonable conditions it is automatically satisfied, and thus eliminate it. To this end consider the following system of mappings. For each n and each $\lambda \in \Lambda_n$ let φ_λ be a mapping (not depending on n) from K_λ to M_λ. Assume that each of the φ's is a normed linear mapping from K_λ to M_λ, that is, φ_λ satisfied (a), (b), and

(c′) $$\|\varphi \circ u\| \leqq m \|u\| \qquad \text{for some } m.$$

Such a system will be denoted by ψ_n^*, and the sequence $\{\psi_n^*\}$ for $n = 0, 1, \cdots$ by ψ^*. The set of all ψ_n^*'s will be denoted by Ψ_n^* and the set of ψ^*'s by Ψ^*. A system ψ_n^* in which the φ_λ's satisfy (a) through (f) will be denoted by ψ_n; similarly for ψ and the corresponding capital letters. The proof that every $\psi \in \Psi$ satisfies (g), so that it defines a decision function, is an immediate consequence of the following lemma. Let $\mathfrak{X}$, L_λ, M_λ and Δ_λ be as before, and for simplicity omit the subscript λ.

LEMMA 1. *Let K be a vector lattice of numerical functions defined on a set Δ. Assume that*

(h) *K is separable for the topology defined by the norm $\|u\| = \sup_{t \in \Delta} |u(t)|$, and*

(k) *there exists a σ-field $\mathfrak{B}$ on Δ such that every positive normed linear functional θ on K can be represented by $\theta \circ u = \|\theta\| \int_\Delta u(t)\, dF(t)$, where F is a σ-additive probability distribution on $(\Delta, \mathfrak{B})$.*

Then every linear mapping φ from K to M satisfying (a), (b), (c), and (d) *can be represented in an essentially unique manner by*

$$\varphi \circ u = \nu_\varphi \text{ equivalence class of } \int_\Delta u(t)\, dF_x(t),$$

where F_x is for each $x \in \mathfrak{X}$ a probability measure on $(\Delta, \mathfrak{B})$.

This lemma is a modified version of Theorem 3 of Doob [6]. (See also Gelfand [7], part 2, par. 7, Thm. 1.) If Δ is locally compact, metrisable, and denumerable at infinity, then the space K of continuous functions with compact nucleus satisfies both (h) and (k). The separability of K is well known and (k) follows immediately from the Riesz representation theorem. For other spaces to which the conclusion of the lemma applies see [6] and the theory of the Daniell integral (for instance Saks [8], p. 328).

Since conditions (a) to (f) are preserved under convex combinations, we have

THEOREM 1. *Under Assumptions 1 to 4, every system $\psi \in \Psi$ of mappings satisfying* (a) *through* (f) *can be obtained in an essentially unique way by a decision procedure, and conversely. The space $\mathfrak{D}$ of decision procedures satisfying Assumptions 1, 3, and 4 can thus be identified with the convex subset Ψ of the linear system Ψ^*.*

Since Ψ^* is a linear space, it is possible to define on it a number of interesting topologies. One such topology $\mathfrak{I}$ is defined below. Under the conditions (3.1)–(3.6) of [1] the topology $\mathfrak{I}$ coincides with the topology of regular convergence of [1]. Consider mappings φ_λ satisfying (a), (b), and (c'). On the set of such mappings we can define a topology $\mathfrak{I}_\lambda$ by the generic neighborhoods

$$V\{\varphi_\lambda^0 \mid \lambda, u, f, \epsilon\} = \{\varphi_\lambda: \ |f\circ\varphi_\lambda\circ u - f\circ\varphi_\lambda^0\circ u| < \epsilon, \ |f\circ\varphi_\lambda\circ a - f\circ\varphi_\lambda^0\circ a| < \epsilon\},$$

where u ranges through K_λ and f through L_λ, and ϵ is a positive number. On Ψ_n^* define the topology $\mathfrak{I}_n$ by replacing φ_λ by ψ_n^* in the preceding formula and letting λ run through Λ_n. For $m < n$ consider the projection $\psi_n^* \to \Pi_{m,n}\psi_n^*$ defined in the following way. If $\lambda \in \Lambda_m$, then to φ_λ corresponds φ_λ. If $\lambda = \{i_1, \cdots, i_m, i_{m+1}, \cdots, i_r\}$, with $m < r \leqq n$, then to φ_λ corresponds $\varphi_{\lambda'}$ with $\lambda' = \{i_1, \cdots, i_m\}$.

It is clear that $\Pi_{mn} = \Pi_{m\nu}\Pi_{\nu n}$ for $m < \nu < n$ and that the projections Π_{mn} are continuous for the topologies $\mathfrak{I}_m$ and $\mathfrak{I}_n$. Let $\mathfrak{I}$ be the topology defined on Ψ^* as projective limit of the $\mathfrak{I}_n$'s with system of projection the $\Pi_{m,n}$'s. (See, for instance, Lefschetz [9], p. 31.) For $\mathfrak{I}$, the linear system Ψ^* becomes a locally convex, Hausdorff vector space. Moreover, conditions (a) through (f) are preserved by passages to the limit under $\mathfrak{I}$; hence Ψ is a closed convex subset of Ψ^*.

4. Compactness of the space of decision functions and lower semicontinuity of the risk functions. This section gives a necessary and sufficient condition that a subset D of $\mathfrak{D}$ be compact in the topology $\mathfrak{I}$ defined above. The word compact is used in the sense that every covering by open sets has a finite subcovering, *not* in the sequential sense. The conditions for compactness of D result from

LEMMA 2. *Let λ be fixed and let Φ be a family of mappings from K_λ to M_λ which satisfy conditions* (a) *through* (d). *Let Assumptions* 1 *to* 4 *be satisfied.*

Then a necessary condition that Φ be relatively compact for $\mathfrak{I}_\lambda$ is that, whatever $f \in L_\lambda^+$ and whatever $\epsilon > 0$, there exists $u(f\circ\epsilon) \in K_\lambda^+(1)$ such that

$$f\circ[\nu_\varphi - \varphi\circ u(f, \epsilon)] \leqq \epsilon \|f\| \qquad \text{for every } \varphi \in \Phi.$$

A sufficient condition is that the preceding condition holds for every f in a subfamily of $\{P_\omega^\lambda\}$ generating the same L-space.

PROOF. If instead of the smallest L-space containing the $\{P_\omega^\lambda\}$ we had taken another space $\mathfrak{L}_\lambda$, then in the statement of the lemma P_ω^λ would have to be replaced by $f \in \mathfrak{L}_\lambda^+(1)$. Lemma 2 is of the same general nature as the well known Helly compactness theorem. The proof sketched below differs from the usual proofs of Helly's theorem in that the denumerable Helly selection principle has been replaced by the topological tool known variously as "ultrafilter" ([10], pp. 25, 59) or "universal net" [11]. The use of this tool is necessary to avoid separability assumptions.

The necessity of the condition is quite obvious. The proof of sufficiency follows. Let U be an ultrafilter on Φ (or a universal net of φ's). Then $\lim_U f\circ\varphi\circ u = \alpha(f, u)$ exists for every $f \in L_\lambda$ and $u \in K_\lambda$. Conditions (a) through (d) are preserved when taking limits so that $\alpha(f, u)$ can be written $f\circ\varphi^0\circ u$ where φ^0 is a mapping having

the properties (a) through (d). Moreover $\lim_U \nu_\varphi = \nu \geqq \nu_{\varphi^0}$. For a given $f \varepsilon \{P_\omega\}$ we have

$$f\circ[\nu_\varphi - \varphi\circ u(f, \epsilon)] < \epsilon, \qquad f\circ[\nu - \varphi^0\circ u(f, \epsilon)] < \epsilon.$$

This implies $f\circ[\nu - \nu_{\varphi^0}] \leqq \epsilon$, hence $f\circ[\nu - \nu_{\varphi^0}] = 0$ for every $f \varepsilon \{P_\omega^\lambda\}$. Consequently $g\circ[\nu - \nu_{\varphi^0}] > 0$ only if g has a nonzero part disjoint from every $f \varepsilon \{P_\omega^\lambda\}$. But then g does not belong to the minimal L-space containing $\{P_\omega^\lambda\}$. This implies $\nu = \lim_U \nu_\varphi = \nu_{\varphi^0} = \nu(\lim_U \varphi)$ so that U converges to φ^0 for the topology $\mathfrak{I}_\lambda$. This leads us to the following

ASSUMPTION 7. A set D of decision functions satifies Assumption 7 if, for each $\epsilon > 0$, each $\lambda \varepsilon \Lambda$, and each $f \varepsilon \{P_\omega^\lambda\}$, there exists a $u \varepsilon K_\lambda$ satisfying

(i) $0 \leqq u \leqq 1$ and $u(t) = 0$ for $t \varepsilon J_\lambda$,

(ii) $f\circ\varphi_\lambda(\delta)\circ u \geqq f\circ\varphi_\lambda(\delta)\circ a_\lambda - \epsilon$,

this last condition being fulfilled uniformly for every $\delta \varepsilon D$.

According to Lemma 2, in order that a set $D \subset \mathfrak{D}$ be compact for $\mathfrak{I}$ it is necessary and sufficient that it be closed and satisfy Assumption 7.

We can now obtain a relation between $\mathfrak{I}$ and the risk function. Let λ be a particular element of Δ_λ and let v be a lower semicontinuous function defined on Δ_λ and such that $v \geqq 0$. If φ is a mapping having the properties (a) through (d), it can be extended to a class of Baire functions by the Daniell-Bourbaki procedure giving

$$\varphi\circ v = \sup_{u \varepsilon K_\lambda; u \leqq v} \varphi\circ u,$$

the supremum being taken in the sense previously defined for measurable functions in $\mathfrak{M}_\lambda$. Under conditions (a) through (d), and assuming $f \varepsilon L_\lambda^+$, the following equality holds

$$f\circ\varphi\circ v = \sup_{u \varepsilon K_\lambda; u \leqq v} f\circ\varphi\circ u = f\circ\Big[\sup_{u \varepsilon K_\lambda; u \leqq v} \varphi\circ u\Big].$$

Thus for the topology $\mathfrak{I}_\lambda$ the function $f\circ\varphi\circ v$ is a lower-semicontinuous function of φ. The same property still holds if v is bounded from below (that is $\inf_{t \varepsilon \Delta_\lambda} v(t) > -\infty$) instead of being nonnegative.

The risk function $R(\omega, \delta)$ is a sum of two terms, $R(\omega, \delta) = R_1(\omega, \delta) + R_2(\omega, \delta)$, where $R_1(\omega, \delta)$ is equal to zero if the probability that the process terminates is unity, and to infinity otherwise, and $R_2(\omega, \delta)$ is the amount paid when the process terminates. Since $R_2(\omega, \delta)$ is a sum of terms of the form $f\circ\varphi_\lambda\circ v$, and since by Assumptions 5 and 6 only a finite number of these terms can be negative, it is clear that $R_2(\omega, \delta)$ is lower semicontinuous on $\mathfrak{D}$ for the topology $\mathfrak{I}$. Moreover, the probability of taking at least n steps is a continuous function of δ so that, by Assumption 5, $R(\omega, \delta)$ is also lower semicontinuous on $\mathfrak{D}$. It is also clear that the assumption of lower semicontinuity in (iii) of Assumption 6 cannot be weakened if $R(\omega, \delta)$ is to be lower-semicontinuous on the whole of $\mathfrak{D}$.

The results just established are collected in

THEOREM 2. *Let $\mathfrak{D}$ be the family of decision functions satisfying Assumptions 1 to 4. If Assumptions 5 and 6 are also satisfied, and D is a subset of $\mathfrak{D}$, then*

(1) *$\mathfrak{D}$ is a closed convex subset of the locally convex Hausdorff vector space Ψ^* supplied with the topology $\mathfrak{z}$. The risk function $R(\omega, \delta)$ is a lower semicontinuous linear function of $\delta \in \mathfrak{D}$ for each $\omega \in \Omega$. Furthermore,*

$$\inf_{\delta \in \mathfrak{D}} R(\omega, \delta) > -\infty \qquad \text{for each } \omega \in \Omega.$$

(2) *In order that D be relatively compact in $(\mathfrak{D}, \mathfrak{z})$, it is necessary and sufficient that D satisfy also Assumption 7. In this case the closure $\bar{D}$ and the closed convex hull $\tilde{D}$ of D in $\mathfrak{D}$ are compact in $(\mathfrak{D}, \mathfrak{z})$.*

In view of Theorem 2, the problem of finding complete classes of solutions can be reduced to similar problems considered by Karlin [2] in an abstract setting. There is, however, some difficulty due to the fact that the risk function $R(\omega,\delta)$ can take strictly infinite values.

5. Complete classes of decision functions. Let A and B be two nonempty sets. In this section A will represent the "states of nature" and B the set of decision functions available to the statistician. Let R be an extended numerical function defined on $A \times B$. Let $H = \{x_1, \cdots, x_k\}$ be a finite subset of A, and let $\{\beta_i\}$ for $i = 1, 2, \cdots, k$ be k strictly positive real numbers such that $\sum_1^k \beta_i = 1$. Let $g = \{(x_i, \beta_i)\}$ for $i = 1, \cdots, k$. Denote by $K(g, y)$ the function $K(g, y) = \sum_1^k \beta_i R(x_i, y)$. For any subset A' of A let $G_{A'}$ be the set of g's corresponding to finite subsets of A'. If H is a finite subset of A, let G_H^+ denote the subset of G_H for which all the β_i's are strictly positive numbers.

The theorems on the existence of complete classes given in the present paper are proved in two steps. It is first shown that a result is correct when the set of "states of nature" is reduced to a finite subset of A or G_A. The result is then extended to an arbitrary A by a limiting process, using the fact that the family Σ of finite subsets of A is directed and covers A. That is, if S_1 and S_2 belong to Σ, then there exists an $S_3 \in \Sigma$ such that $S_1 \cup S_2 \subset S_3$ and moreover $A \subset \bigcup_{S \in \Sigma} S$. The required tool for such a passage to the limit is given by the following entirely obvious proposition.

LEMMA 3. *Let u be an extended numerical function defined on G_A. Let G be a subset of G_A such that, for every $y \in B$, the inequality $K(g, y) \leqq u(g)$ for every $g \in G$ implies $K(g, y) \leqq u(g)$ for every $g \in G_A$. Let Σ be a directed family of subsets of G_A covering G. Finally let C be a subset of B having properties:*

(C_1) C is supplied with a topology for which it is a compact topological space (in the Borel-Lebesgue sense);

(C_2) for every fixed $g \in G$ the function defined by $y \to K(g, y)$ is lower semicontinuous on C;

(C_3) for every $S \in \Sigma$ and every $\epsilon > 0$, there exists a $y_\epsilon \in C$ such that $K(g, y_\epsilon) \leqq u(g) + \epsilon$ for every $g \in S$.

Then, there exists $y \in C$ such that $K(g, y) \leqq u(g)$ for every $g \in G_A$.

A particular family Σ used below is obtained as follows. For a finite subset H

of A, let S be a convex subset of G_H^+ spanned by a finite number of points of G_H^+. Let G_H be supplied with the usual Euclidean metric, and assume moreover that S is closed in G_H for this metric. It is clear that the family Σ_H of such subsets of G_H^+ is directed and covers G_H^+.

Let y_0 be a particular point of B and assume that $K(g, y_0)$ is finite for every $g \in G_H^+$. It is also clear that $K(g, y_1) \leqq K(g, y_0)$ for every $g \in G_H^+$ implies $K(g, y_1) \leqq K(g, y_0)$ for every $g \in G_H$, provided that $K(g, y_1)$ is bounded from below on G_H.

In order to obtain completeness theorems, it will be sufficient to add, to the preceding lemma, the following result.

LEMMA 4. *Let H be a finite subset of A, say $H = \{x_1, \cdots, x_k\}$. Assume that B is convex with respect to (H, R), that is, whatever may be $y_1, y_2 \in B$ and $0 \leqq \alpha \leqq 1$, there exists $y \in B$ such that*

$$R(x, y) \leqq \alpha R(x, y_1) + (1 - \alpha)R(x, y_2), \qquad \text{for every } x \in H.$$

Assume moreover that $\inf_{y \in B} R(x_i, y) > -\infty$ for each $x_i \in H$. Let $a = \inf_{y \in B} \sup_{x \in H} R(x, y)$.

Then, whatever $b < a$, there exists $g \in G_H^+$ such that $K(g, y) > b$ for every $y \in B$.

Or, equivalently, if b' is such that whatever $g \in G_H^+$ there exist $y_g \in B$ such that $K(g, y_g) \leqq b'$, then whatever $a' > b'$ there exists $y \in B$ such that $K(g, y) < a'$ for every $g \in G_H$.

This lemma is well known for the case where $R(x, y)$ takes only finite values (see for instance Karlin [2]). For a proof readily adaptable to our case see Ville [12] and Kneser [13]. The lemma is obvious if H contains only two elements, and an extension by induction does not present any difficulties. We will make use of the following assumptions.

ASSUMPTION 8. For every $x \in A$, $\inf_{y \in B} R(x, y) > -\infty$.

ASSUMPTION 9. Whatever may be $y_1, y_2 \in B$ and $0 \leqq \alpha \leqq 1$, there exists $y \in B$ such that $R(x, y) \leqq \alpha R(x, y_1) + (1 - \alpha)R(x, y_2)$.

ASSUMPTION 10. B is a compact topological space and $R(x, y)$ for each x is lower semicontinuous on B.

It is well known and obvious that these assumptions imply the completeness of the class of admissible solutions.

Let $h(g) = \inf_{y \in B} R(g, y)$ and let G_A^* be the subset of G_A for which $h(g) < +\infty$.

THEOREM 3. *Let Assumptions 8, 9, and 10 be satisfied. If $y_0 \in B$ is such that $\inf_{g \in G_A^*} K(g, y_0) - h(g) \geqq \epsilon$ then there exists $y_1 \in B$ such that $K(g, y_1) + \frac{1}{2}\epsilon \leqq K(g, y_0)$ for every $g \in G_A$. Hence the class of Bayes' solutions in the wide sense is complete.*

PROOF. It results from Lemma 4 that such a y exists for every finite subset of G_A^*. Consequently by Lemma 3 there exists a y_1 such that the condition be satisfied for every $g \in G_A$. This result is related to a result of Kiefer [14].

If Assumptions 1 to 6 are satisfied and the set D of decision functions considered is compact, Theorem 3 implies that the class of Bayes' solutions in the wide sense is complete. To prove this result, Kiefer used the boundedness of the loss functions.

If G is a convex subset of G_A supplied with some topology for which it is compact, and $K(g, y)$ is lower-semicontinuous for every $y \in B$, and moreover such that $h(g)$ is upper-semicontinuous on G, then $K(g, y) - h(g)$ must reach its minimum on G. For such subsets G, Bayes' solutions in the wide sense coincide with Bayes' solutions. This applies in particular to the sets $S \in \Sigma_H$ introduced after the statement of Lemma 3. Take then for C in Lemma 3 the closure of the class of Bayes' solutions and apply Lemma 3 twice, first with $\Sigma = \Sigma_H$ for each H and then for $\Sigma = \{G_H^+\}$, with H finite. This gives

THEOREM 4. *If assumptions* (8), (9), *and* (10) *are satisfied, the closure of the class of Bayes' solutions is a complete class.*

Let G_Ω be the set of a priori distributions with finite support on Ω. Let $g = \{\{\omega_i\},\ i = 1 \cdots k;\ \{\alpha_i\},\ i = 1 \cdots k\}$ be an element of G_Ω. Denote by $K(g, \delta)$ the corresponding risk function $K(g, \delta) = \sum_1^k \alpha_i R(\omega_i, \delta)$. From Theorem 4 we obtain at once

THEOREM 5. *Let Assumptions* 1 *to* 6 *be satisfied and let S be a convex subset of G_Ω. Let D be a class of decision functions satisfying Assumption* 7 *and let $\bar{D}$ be its closed convex hull. Let $B \subset \bar{D}$ be the class of decision functions δ satisfying $\delta \in \bar{D}$ and such that $K(g, \delta) = \min_{\delta' \in \bar{D}} K(g, \delta')$ for some $g \in S$. Let $\bar{B}$ be the closure of B. Let C be the class of δ's satisfying*

$$\delta \in \bar{D}, \qquad \inf_{g \in S} [K(g, \delta) - \min_{\delta' \in \bar{D}} K(g, \delta')] = 0.$$

Then, whatever may be $\delta_0 \in \bar{D}$, there exists $\delta_1 \in \bar{B} \cap C$ such that

$$K(g, \delta_1) \leqq K(g, \delta_0) \qquad \text{for every } g \in S.$$

REMARK. The preceding theorem is an extension of Wald's Theorems 3.17 and 3.18 ([1], p. 100). Usually the class S can be replaced by a convex class of a priori distributions which do not necessarily have finite support. Moreover, an inequality of the type $K(g, \delta_1) \leqq K(g, \delta_0)$ for every $g \in S$ usually implies that the same inequality holds for a larger class.

This brings us close to Wald's Theorem 3.19 ([1], p. 101). It is, however, clear that Wald meant to assume the convexity of the class ζ of a priori distributions used in this theorem. Wald's proof uses the convexity of ζ; it is easy to show on examples that without this assumption the conclusion of the theorem does not generally hold. This correction being made, it is possible to use our Theorem 5 to obtain extensions of Wald's Theorem 3.19.

6. Miscellaneous remarks.

(1) The theorems given in this paper are generalizations of Wald's [1] Theorems 3.17 and 3.18. Assumptions on Ω are required to obtain the equivalent of his Theorem 3.20. The same is true of Wolfowitz's [15] theorems on ϵ-complete classes.

(2) Wald [1] assumed that the spaces L_n of equivalence classes of integrable functions on $(\mathfrak{X}, \mathfrak{A}_n, \mu_n)$ are separable. In this case our Assumption 2 is certainly satisfied. Moreover, the topology $\mathfrak{I}$ on $\mathfrak{D}$ admits at each point a countable basis,

so that a compact subset of $\mathfrak{D}$ is also sequentially compact. Consequently every admissible decision function is a limit of a *sequence* of Bayes' decision procedures.

In this case it is possible to use general a priori distributions on Ω instead of the discrete distributions considered in the present paper since, according to Fatou's lemma, $\int R(\omega, \delta)\, d\xi(\omega)$ is lower semicontinuous in δ.

(3) In many problems the spaces $(\mathfrak{X}, \mathfrak{A})$ are countable discrete spaces. In such cases the method used here is unnecessarily involved. The topologies $\mathfrak{I}_\lambda$ become topologies of pointwise convergence of the measures $\varphi(\lambda, x, \delta)$.

Thus the separability of K_λ is no longer necessary. For instance, it is possible to choose for T_λ any compact space, or even any completely regular space, and then to let K_λ be, for instance, the space of all continuous functions on Δ_λ.

(4) We will now give another example of application of Lemma 3. Suppose that Assumptions 1 to 6 are satisfied and assume that T_λ and $W(\omega, t; \lambda)$ do not depend on λ. It frequently happens that under such conditions the statistician would be able to find probability measures $\delta(x, \lambda)$ defined on T such that

$$r(\omega, \delta, \lambda) = E\{\delta(x, \lambda) \circ W(\omega, t) \mid \omega\}$$

is finite for each $\omega \in \Omega$, provided only that λ is large enough, say provided that the number of elements in λ be larger than some $n(\omega)$. It is clear that under circumstances, whatever may be the finite set $\{\omega_1, \cdots, \omega_k\}$ and whatever $\epsilon > 0$ and whatever may be the decision function δ_0, there exists a finite N and a decision function δ_N terminating surely in not more than N steps such that

$$R(\omega_i, \delta_N) \leqq R(\omega_i, \delta_0) + \epsilon \quad \text{for every} \quad \omega_i \in \{\omega_1, \cdots, \omega_k\}.$$

If we assume that T is compact this implies that there exists a decision function δ_1 which is Bayes' among those terminating in not more than N steps and which satisfies

$$R(\omega_i, \delta_1) \leqq R(\omega_i, \delta_0) + 2\epsilon, \qquad i = 1, 2, \cdots, k.$$

According to Lemma 3 the closure of the class of solutions which for some N are Bayes' among those requiring less than N steps is then complete. This has immediate applications to sequential analysis.

(5) The assumption of compactness used in Lemma 3, Theorem 3, and Theorem 4 is obviously too strong. With the notation of Lemma 3, let $\mathfrak{F}$ be the space of all extended numerical functions on G_A topologized by the topology of pointwise convergence on G_A. The space $\mathfrak{F}$ is a compact Hausdorff space for this topology. For each $y \in C$ let W_y by the function of g defined by $g \to K(g, y)$ and let F be the family $F = \{W_y ; y \in C\} \subset \mathfrak{F}$. Furthermore, let $\bar{F}$ be the closure of F in $\mathfrak{F}$. The only property actually used in the proofs of Lemma 3, Theorem 3 and Theorem 4 is that for every $v \in \bar{F}$ there exists a $W \in F$ such that $W(y) \leqq v(g)$ for every $g \in G_A$. This property is the analogue of the property of "weak compactness" used by Wald ([1] p. 53); it will be referred to as property (W).

Similarly in Theorem 5 the closure $\bar{B}$ of the class of Bayes' procedures could

be replaced by any class B^* containing B and having the property (W). It is easy to see that whenever $\tilde{D}$ satisfies (W) the same is also true of the class of admissible solutions of $\tilde{D}$, although these classes might not be compact.

(6) In many problems it is convenient to compact the spaces T_λ instead of restricting the set of decision functions to a compact subset of $\mathfrak{D}$. For instance, let T_λ be precompact and let its compacted $\bar{T}_\lambda$ be metrisable. Furthermore, assume that for every sequence $\{t_n\} \subset T_\lambda$ converging to a point $t_0 \varepsilon \bar{T}_\lambda - T_\lambda$ the following equality holds

$$\liminf_{n\to\infty} W(\omega, t_n; \lambda) = \sup_{t \varepsilon T_\lambda} W(\omega, t; \lambda).$$

Take for each (ω, λ) the largest lower semicontinuous extension of $W(\omega, t, \lambda)$ on $\bar{T}_\lambda$. These Assumptions 1 to 7 are satisfied so that the set $\tilde{\mathfrak{D}}$ of measurable decision functions obtained by replacing T_λ by $\bar{T}_\lambda$ is compact. However, if for instance T_λ is a Borel subset of $\bar{T}_\lambda$, every $\delta_0 \varepsilon \mathfrak{D}$ is dominated by some $\delta_1 \varepsilon \mathfrak{D}$. In this case $\mathfrak{D}$ has the property (W) without being compact.

As a simple example consider the case where T is for each λ the whole real line R. Then $\bar{T}_\lambda$ can be taken as the extended real line (compacted by the adjunction of points at infinity). Let ω be a real valued parameter and let $W(\omega, t; \lambda) = (\omega - t)^2$. Then the class $\mathfrak{D}$ is not compact but $\tilde{\mathfrak{D}}$ is compact and every $\tilde{\delta} \varepsilon \tilde{\mathfrak{D}}$ is dominated by the δ obtained by replacing infinite values of the estimate by say zero. The preceding argument gives practical content to a remark of Karlin [3] that "every game can be completed to a game having a value."

(7) After this paper was written, an article by M. N. Ghosh [16] was brought to the author's attention. Ghosh's assumptions are stronger than those of the present paper. For instance, his Assumptions IV and VII imply Assumption 3 of the present paper, and his Assumptions III and IV are substantially stronger than our Assumption 6. Ghosh's theorems on compactness and existence of complete classes are special cases of Theorems 2 and 5 of the present paper.

(8) To give a simple illustration of the relevance of the results of the present paper, consider the problem of estimating the mean of a normal population with known variance from a sample of fixed size. Take T and Ω to be the real line. Let $W(\omega, t)$ be any positive nondecreasing function of $\tau = |\omega - t|$, continuous on the left and such that $\lim_{\tau\to\infty} W(\tau) = \infty$. For instance $W(\omega, t)$ can be taken equal to $(\omega - t)^2$. Then

(a) Wald's intrinsic topology on T is usually the discrete topology on T.

(b) If W is not continuous the assumptions of [16] are not necessarily satisfied.

(c) According to Theorem 2 the subclass of decision functions satisfying $R(\omega_0, \delta) \leqq M < \infty$ for one arbitrary given ω_0 is a closed convex compact subset of $\mathfrak{D}$.

(d) While $\mathfrak{D}$ itself is not compact the class of estimates $\tilde{\mathfrak{D}}$ obtained by allowing infinite values for t is compact in the corresponding topology. According to Remark (6), $\mathfrak{D}$ satisfies also property (W).

(e) The class B^* obtained by taking the closure in $\tilde{\mathfrak{D}}$ of the class of Bayes'

solutions and then replacing infinite values of the estimate by zero or even by the value of any finite estimate, is a complete class for $\mathfrak{D}$.

Instead of estimating the mean ω by a point estimate, it might be desirable to give confidence intervals for ω. For instance, it might be desirable to obtain confidence intervals of given length $2l$. In such a case take $W(\omega, t)$ to be zero if $|\omega - t| \leqq l$ and unity if $|\omega - t| > l$. Again allow t to take infinite values. The Assumptions 1 to 7 are then satisfied so that Theorem 5 applies directly to $\mathfrak{D}$.

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SEQUENTIAL LIFE TESTS IN THE EXPONENTIAL CASE[1]

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1. Introduction and summary. This paper describes sequential life test procedures, considering, as in a recent paper [4] devoted to nonsequential methods, the special case in which the underlying distribution of the length of life is given by the exponential density

$$f(x, \theta) = e^{-x/\theta}/\theta, \qquad x > 0. \tag{1}$$

The unknown parameter $\theta > 0$ can be thought of physically as the mean life.

Our primary aim is to test the simple hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1$, where $\theta_1 < \theta_0$, with type I and II errors equal to preassigned values α and β, respectively. The test is carried out by drawing n items at random from the population and placing them all on a life test. We consider both the replacement case, in which failed items are immediately replaced by new items, and the nonreplacement case.

The test can be terminated either at failure times with rejection of H_0, or at any time between failures with acceptance of H_0. Since abnormally long intervals between failures furnish "information" in favor of H_0 and abnormally short intervals furnish "information" in favor of H_1, these features are not only reasonable but actually desirable. Similar problems involving a continuous time parameter have recently appeared [3], [5].

In this paper we obtain likelihood ratio tests and give approximate formulae for the O.C. (operating characteristic) curve, for the expected number of failures $E_\theta(r)$, and for the expected waiting time $E_\theta(t)$ before a decision is reached. In the replacement case where the number of items on test throughout the experiment is the same, namely n, it is shown that $E_\theta(t) = (\theta/n)E_\theta(r)$. A table giving approximate values of $E_\theta(r)$ for certain choices of θ_0/θ_1, α, and β is given for the replacement case. Some calculations of exact $L(\theta)$ and $E_\theta(r)$ values using formulae in [1] and [3] are reported. Several numerical examples are worked out.

2. Basic formulae. Wald's work on sequential analysis [8] can be used virtually without modification in a situation where decisions are made continuously. In fact, in a truly continuous situation, Wald's formulae become exact, since there is then no excess over the boundary. It will become clear as we proceed that, in the problem at hand, the situation can be termed semicontinuous (not to be confused with the concept of the same name in real variable theory). There is no excess over the boundary used for accepting H_0, but in general there will be some excess over the boundary used in accepting H_1.

Let us assume that from the underlying exponential p.d.f. (1), n items are

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drawn at random and placed on life test. We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ with type I error $= \alpha$ and type II error $= \beta$. Since information is available continuously, a continuous analogue of the sequential probability ratio test of Wald can be used. The decision as time unfolds depends on

$$B < (\theta_0/\theta_1)^r \exp[-(1/\theta_1 - 1/\theta_0)V(t)] < A \tag{2}$$

where B and A are constants, depending on α and β, such that $B < 1 < A$. The decision to continue experimentation is made as long as the inequality (2) holds. At the time the experiment is stopped, if the first inequality in (2) is violated, we accept H_0 ; if the second inequality is violated, we accept H_1. As in Wald's case, the test obtained by setting $B = \beta/(1 - \alpha)$ and $A = (1 - \beta)/\alpha$ is a satisfactory solution of the problem from a practical point of view. Details are given in remarks 1 and 2 in Section 4.

In (2), $V(t)$ is a statistic which can be interpreted as the total life observed up to time t. In the replacement case

$$V(t) = nt, \tag{3}$$

while in the nonreplacement case[2]

$$\begin{aligned} V(t) &= \sum_{i=1}^{r} (n - i + 1)(x_i - x_{i-1}) + (n - r)(t - x_r) \\ &\equiv \sum_{i=1}^{r} x_i + (n - r)(t - x_r), \end{aligned} \tag{4}$$

where x_i denotes the time of the ith failure, with $x_0 = 0$.

To graph the data continuously in time, it is convenient to write (2) in the form

$$-h_1 + rs < V(t) < h_0 + rs, \tag{5}$$

where h_0 , h_1 , and s are positive constants given by

$$h_0 = \frac{-\log B}{1/\theta_1 - 1/\theta_0}, \qquad h_1 = \frac{\log A}{1/\theta_1 - 1/\theta_0}, \qquad s = \frac{\log (\theta_0/\theta_1)}{1/\theta_1 - 1/\theta_0}. \tag{6}$$

Further, it can be shown ([8], pp. 48–50) that the O.C. curve, that is, the probability of accepting H_0 when θ is the true parameter value, is given approximately by a pair of parametric equations

$$L(\theta) = \frac{A^h - 1}{A^h - B^h}, \qquad \theta = \frac{(\theta_0/\theta_1)^h - 1}{h(1/\theta_1 - 1/\theta_0)}, \tag{7}$$

by letting the parameter h run through all real values.

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[2] In the nonreplacement case it may happen that no decision has been reached by the time $t = x_n$, when all n items have failed. This will then require that we either put more items on test and wait until (2) is violated or else have a rule which will tell us how to terminate the experiment and with what decision at $t = x_n$. Fortunately n is often at our disposal and so can be chosen sufficiently large so that the probability of reaching no decision by time x_n is negligible. For large enough n, it really makes very little difference how we truncate experimentation. We could, for example, adopt the rule that H_1 is accepted if (2) is satisfied for all $t \leq x_n$.

The values of $L(\theta)$ at the five points $\theta = 0, \theta_1, s, \theta_0$, and ∞ enable one to sketch the entire curve. These values are respectively 0, β, $\log A/(\log A - \log B)$, $1 - \alpha$, and 1.

We now give, in terms of $L(\theta)$, an approximate formula for $E_\theta(r)$, the expected number of observations required to reach a decision when θ is the true parameter value. Since the logarithm of the middle expression in (2) is either $\log B$ or $\log A$ at the time experimentation stops, we have, neglecting only the excess over $\log A$,

$$E_\theta(r) \log (\theta_0/\theta_1) - E_\theta(V(t))[1/\theta_1 - 1/\theta_2] \sim L(\theta) \log B + [1 - L(\theta)] \log A. \tag{8}$$

It is proved in the next section that

$$E_\theta(V(t)) = \theta E_\theta(r). \tag{9}$$

Hence we have from (8) and (9)

$$E_\theta(r) \sim \begin{cases} \dfrac{L(\theta) \log B + [1 - L(\theta)] \log A}{\log (\theta_0/\theta_1) - \theta(1/\theta_1 - 1/\theta_0)} = \dfrac{h_1 - L(\theta)(h_0 + h_1)}{s - \theta}, & \theta \neq s \\ \dfrac{-\log A \quad \log B}{[\log (\theta_0/\theta_1)]^2} = \dfrac{h_0 h_1}{s^2}, & \theta = s. \end{cases} \tag{10}$$

If we let $k = \theta_0/\theta_1$, the approximate values of $E_\theta(r)$ become particularly simple when $\theta = \theta_1, s$, or θ_0. They are

$$\begin{aligned} E_{\theta_1}(r) &\sim [\beta \log B + (1 - \beta) \log A] \,/\, [\log k - (k - 1)/k], \\ E_s(r) &\sim -\log A \quad \log B \,/\, (\log k)^2, \\ E_{\theta_0}(r) &\sim [(1 - \alpha) \log B + \alpha \log A] \,/\, [\log k - (k - 1)]. \end{aligned} \tag{11}$$

In Table 1 we give $E_\theta(r)$ for five values of θ $(0, \theta_1, s, \theta_0, \infty)$, for four values of k $(3/2, 2, 5/2, 3)$, and for the four number pairs (α, β) which can be made with the numbers .01 and .05.

3. A basic identity. In this section (9) is derived. While this result can be obtained as a consequence of a theorem of Doob on continuous parameter martingales ([2], p. 376), a simpler proof is desirable. We shall consider the replacement case, where $V(t) = nt$, although the proof can be trivially modified so as to hold in nonreplacement and truncated situations.

In the replacement case (9) becomes

$$E_\theta(t) = E_\theta(r)\, \theta/n. \tag{12}$$

Thus we are relating expected waiting time to reach a decision to the expected number of failures.

To prove (12) we introduce a "large" integer N and let x_N denote the time of the Nth failure. Let t denote the (first) time at which the inequality (2) is violated or the time x_N, whichever comes sooner. Then we can write

$$x_N = t + (x_{r+1} - t) + (x_{r+2} - x_{r+1}) + \cdots + (x_N - x_{N-1}). \tag{13}$$

TABLE 1

Approximate values of $E_\theta(r)$ for sequential tests for various values of $k = \theta_0/\theta_1$ and α, β

$k = \theta_0/\theta_1$		3/2		2		5/2		3	
α		.01	.05	.01	.05	.01	.05	.01	.05
θ	β								
0	.01	11	7	7	4	5	3	4	3
	.05	11	7	7	4	5	3	4	3
θ_1	.01	62.4	40.3	23.3	15.1	14.2	9.20	10.4	6.74
	.05	60.4	36.7	22.6	13.7	13.8	8.38	10.1	6.14
ε	.01	128	82.7	43.9	28.3	25.1	16.2	17.5	11.3
	.05	82.7	52.7	28.3	18.0	16.2	10.3	11.3	7.18
θ_0	.01	47.6	44.2	14.7	13.6	7.71	7.16	5.00	4.63
	.05	30.8	28.0	9.48	8.64	4.99	4.54	3.23	2.94
∞	any	0	0	0	0	0	0	0	0

Three actions are possible: accept H_1 before x_N, in which case $t = x_r$; accept H_0 before x_N, in which case $x_r < t < x_{r+1}$; or take no action before x_N, in which case $t = x_N$ and $r = N$. Since N is fixed in advance

$$E(x_N) = N\theta/n. \tag{14}$$

Further, it is easily verified that the $(N - r)$ random variables $(x_{r+1} - t)$, $(x_{r+2} - x_{r+1}), \cdots, (x_N - x_{N-1})$ are independently and identically distributed with the exponential density $(n/\theta)e^{-nx/\theta}$, for $x > 0$. Hence if we take the expectation of both sides of (14), first holding r fixed and then taking the expectation with respect to r, we obtain

$$N\theta/n = E_\theta(t; N) + [N - E_\theta(r; N)]\theta/n \tag{15}$$

or

$$E_\theta(t; N) = E_\theta(r; N)\theta/n. \tag{16}$$

Formula (16) holds for all N. As $N \to \infty$ the probability of coming to a decision before x_N tends to unity. Moreover as $N \to \infty$

$$E_\theta(r; N) \uparrow E_\theta(r), \qquad E_\theta(t; N) \uparrow E_\theta(t), \tag{17}$$

where $E_\theta(r)$ and $E_\theta(t)$ are respectively the expected number of failures and expected waiting time to reach a decision if $N = \infty$. Thus it follows, letting $N \to \infty$,

that (16) becomes (12). The nonreplacement case can be treated in exactly the same way, because

$$V(x_n) = V(t) + (n - r)(x_{r+1} - t) + (n - r - 1)(x_{r+2} - x_{r+1}) + \cdots + (x_n - x_{n-1}). \tag{18}$$

As before, $t = x_r$, if H_1 is accepted; $x_r < t < x_{r+1}$, if H_0 is accepted; and $t = x_n$, if no decision is reached by the time all n items have failed. The last $(n - r)$ components on the right side of (18) are mutually independent random variables, each distributed with the p.d.f. (1). Thus it follows as in the replacement case that

$$E_\theta(V(t); n) = \theta E_\theta(r; n). \tag{19}$$

As n increases, $E_\theta(r; n) \uparrow E_\theta(r)$, the expected number of failures in reaching a decision in the replacement case. Thus no matter how we decide to terminate experimentation, $E_\theta(V(t); n)$ can be replaced by $E_\theta(V(t)) = \theta E_\theta(r)$, when n is large. In practice, for "large" n one could take $n > 3 \max_\theta E_\theta(r)$.

While (12) relates expected waiting time to the expected number of failures in the replacement case, (19) relates expected total life (not waiting time) to the expected number of failures in the nonreplacement case. Actually one has to know the probability distribution of r in order to compute $E_\theta(t)$ exactly in the nonreplacement case. It can be shown that in the nonreplacement case the formula for $E_\theta(t)$ is given by

$$E_\theta(t) = \sum_{k=1}^{n} \Pr(r = k \mid \theta) E_\theta(X_{k,n}), \qquad E_\theta(X_{k,n}) = \theta \sum_{i=1}^{k} \frac{1}{n - i + 1}. \tag{20}$$

In the replacement case one has, analogous to (20),

$$E_\theta(t) = \sum_{k=1}^{\infty} \Pr(r = k \mid \theta) E_\theta(X_{k,n}), \qquad E_\theta(X_{k,n}) = \frac{k\theta}{n}. \tag{20'}$$

where n is the sample size maintained throughout the experiment. Thus, in the replacement case (20') clearly becomes (12). Equation (20) is valid for all life test procedures which involve nonreplacement. Similarly (20') holds for all life test procedures, where items which fail are replaced.[3]

4. Some remarks. This section contains three extended remarks on certain aspects of Sections 2 and 3.

Remark 1. Upper and lower bounds for $L(\theta)$ and $E_\theta(r)$. The formulae for $L(\theta)$ and $E_\theta(r)$, given by (7) and (10) respectively, are approximations to the actual $L(\theta)$ and actual $E_\theta(r)$ arising from the use of the semi-continuous sequential decision rule specified by the inequalities (2). The question arises as to how good these approximations are. A modification of the results of Wald on bounds for the

[3] Here, "all" means truncated or untruncated, sequential, or any similar procedures. Of course the probability distribution of r does depend on the procedure which is followed. In [6] explicit formulae for $\Pr(r = k \mid \theta)$ are worked out for three procedures.

O.C. and ASN curves in the binomial case [8] and of results of Herbach [7] on the discrete Poisson yields the following bounds on the actual $L(\theta)$ and $E_\theta(r)$:

$$\frac{A^h - 1}{A^h - B^h} \leqq L(\theta) \leqq \frac{(kA)^h - 1}{(kA)^h - B^h}, \qquad h \neq 0 \text{ (that is, for } \theta \gtrless s), \tag{21}$$

$$\frac{L(\theta)\log B + [1 - L(\theta)]\log A}{\log k - \theta(1/\theta_1 - 1/\theta_0)} \begin{Bmatrix} \leqq \\ \geqq \end{Bmatrix} E_\theta(r) \begin{Bmatrix} \leqq \\ \geqq \end{Bmatrix} \frac{L(\theta)\log B + [1 - L(\theta)][\log A + \log k]}{\log k - \theta(1/\theta_1 - 1/\theta_0)} \tag{22}$$

where the upper inequality signs hold for $\theta < s$ and the lower inequality signs hold for $\theta > s$.

One unpleasant feature of the bounds given in (22) is that they involve $L(\theta)$, which is unknown. However, this matters little in actual practice because the limits on $L(\theta)$ given by (21) are quite close together for the range of values of k and (α, β) covered in Table 1. Thus, for example, for

$$k = \theta_0/\theta_1 = 3, \quad \alpha = \beta = .05, \quad A = (1 - \beta)/\alpha = 19, \quad B = \beta/(1 - \alpha) = 1/19,$$

we get from (21) $.95 \leqq L(\theta_0) \leqq .983$ and $.05 \leqq L(\theta_1) \leqq .052$. The upper and lower bounds for $E_\theta(r)$ given by (22) are close together for $\theta = \theta_0$ and comparatively far apart for $\theta = \theta_1$. Thus for the case $k = 3$ and $\alpha = \beta = .05$, the difference between the upper and lower bounds is $\leqq .06$ for $\theta = \theta_0$ and is about 2.5 for $\theta = \theta_1$.

The left side of (22) is the approximate formula (10) for $E_\theta(r)$ except that the $L(\theta)$ in (22) refers to the exact value and the $L(\theta)$ in (10) is given by the approximation (7). In view of the preceding paragraph, the values of $E_{\theta_0}(r)$ given in Table 1 are very close to the correct values, while the values of $E_{\theta_1}(r)$ are essentially lower bounds for the correct value. We cannot say more unless we go through more extensive calculations of the sort to be described in Remark 2.

Remark 2. Some exact calculations of $L(\theta)$ and $E_\theta(r)$.

Wald ([8], pp. 45–46) pointed out that in order to have a test of exactly strength (α, β), the A and B in (2) should be replaced by A^* and B^*, where $A^* \leqq A = (1 - \beta)/\alpha$ and $B^* \geqq B = \beta/(1 - \alpha)$. In the present case, with information available continuously in time, $B^* = B = \beta/(1 - \alpha)$ since the acceptance of H_0 involves no excess over the boundary. However, acceptance of H_1 does, in general, entail a positive excess over the boundary, and all we can say initially about A^* is that it should lie between $A\ \theta_1/\theta_0$ and A. Thus using $A = (1 - \beta)/\alpha$ instead of A^* is an approximation.

The approximate test based on using A and B is suitable for all practical purposes, since one consequence of the inequalities (21) is that the strength (α', β') is such that $\alpha' \leqq \alpha$, $\beta \leqq \beta' \leqq \beta/(1 - \alpha)$. Since α and β are generally small ($\leqq .10$ say) a procedure based on A and B provides essentially the same protection against errors of the first and second kind as does the test based on using A^* and B^*. However, the use of A rather than A^* in (2) will entail a small increase in $E_\theta(r)$, particularly for $\theta < s$.

As a practical matter, one would usually be content with a test based on (2) which uses A and B. As a matter of fact, this is what is done all the time by people faced with a practical decision problem. For most sequential problems, the problem of finding the A^* and B^* which will give exactly strength (α, β) has not been solved. One has to rely, in such cases, on the results of Wald which indicate that the errors involved in using A, B, and approximate formulae for $L(\theta)$ and $E_\theta(r)$ are "reasonably" small.

In the problem at hand we know, in view of the continuous availability of information, that $B^* = B = \beta/(1 - \alpha)$. Furthermore, formulae are available for computing A^* and for computing O.C. and ASN curves exactly. The formulae for accomplishing these tasks are available [1], [3]. While the computational labor involved in any special case is exceedingly heavy, the results of such computations do throw some light on how exact O.C. and ASN curves compare with those computed by using approximations.

Formulae (4.17) and (4.23) in [3] (similar formulae are given in [1], p. 102) were used to compute

(i) the exact O.C. and $E_\theta(r)$ curves for the semi-continuous rule (2) with $B = \beta/(1 - \alpha)$ and $A = (1 - \beta)/\alpha$. This was done for the case $k = \theta_0/\theta_1 = 3$ and $\alpha = \beta = .05$, and

(ii) A^* (where $A\ \theta_1/\theta_0 \leqq A^* \leqq A$) such that the decision rule

$$(2') \qquad \beta/(1 - \alpha) < (\theta_0/\theta_1)^r \exp[-(1/\theta_1 - 1/\theta_0)\, V(t)] < A^*$$

has an O.C. curve for which $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) = \beta$ exactly, and then to compute $E_\theta(r)$ for the (B, A^*) rule. This was done for the cases $\alpha = \beta = .05$ and $k = \theta_0/\theta_1 = 3/2$, 2, and 3, and also for $\alpha = \beta = .01$ and $k = 3$.

The result of (i) was

$$L(\theta) = .968, \quad L(s) = .529, \quad L(\theta_1) = .051,$$

$$E_{\theta_0}(r) = 3.03, \quad E_s(r) = 8.10, \quad E_{\theta_1}(r) = 7.00.$$

Computation (ii) gave

$\alpha = \beta$	k	A^*	$E_{\theta_0}(r)$	$E_s(r)$	$E_{\theta_1}(r)$
.05	3	13.25	2.94	7.22	6.21
	2	15.1	8.64	18.0	13.8
	3/2	16.6	27.9	52.8	36.8
.01	3	68.9	5.00	17.5	10.5

Bearing in mind that the computations were carried through only in a small number of cases, one can make three observations:

(a) For the case $k = 3$ and $\alpha = \beta = .05$, the use of $B = 1/19$ and $A = 19$ results in getting $\alpha' = .032$ and $\beta' = .051$ as compared with $\alpha = \beta = .05$ when one uses $B^* = B = 1/19$ and $A^* = 13.25$. Also, $E_\theta(r)$ is increased by .09, .88, and .79 at $\theta = \theta_0, s, \theta_1$ respectively.

(b) Of more interest is the fact that the exact values of $E_\theta(r)$ for the (B, A^*) rule practically coincide with the *approximate* values of $E_\theta(r)$ computed for the (B, A) rule by formulae (10) and (11) and given in Table 1.

(c) In the range of values of $k = \theta_0/\theta_1$ and of α and β covered by Table 1, a good guess at the value of A^* is the value A^{**} lying midway between A and A/k, the upper and lower limits on A^*. This means that $A^{**} = (k + 1)A/2k$. On the basis of our limited calculations we conjecture that in the range of values covered in Table 1, a semi-continuous decision rule (2) with A replaced by A^{**} will have almost exactly strength (α, β). The values of $E_\theta(r)$ associated with a (B, A^{**}) rule will be given to a close approximation by (10).

Remark 3. An approximate formula for $E_\theta(t)$ in the nonreplacement case. A useful approximation to $E_\theta(t)$ in the nonreplacement case is given by $E_\theta(t) \sim \theta \log (n/[n - E_\theta(r)])$. This approximation is obtained by replacing $E_\theta(X_{k,n})$ in (20) by its approximation $\theta \log (n/[n - k])$. Thus (20) becomes

$$E_\theta(t) \sim \theta\, E_\theta\left[\log\left(\frac{n}{n-k}\right)\right] \sim \theta \log\left(\frac{n}{n - E_\theta(r)}\right).$$

This approximation has been tested numerically by calculations on truncated nonreplacement decision procedures, where the exact values of $E_\theta(t)$ can be computed and compared with the suggested approximation. The agreement is close.

5. Numerical examples. These methods have been applied to eight problems.

Problem 1. Find a sequential replacement procedure for testing $H_0 : \theta = \theta_0 = 7500$ hours against $H_1 : \theta_1 = 2500$ hours with $\alpha = \beta = .05$. The constant number of items under test is $n = 100$.

Solution 1 (approximate). The (B, A) test (2) becomes in this case

$$\tfrac{1}{19} < 3^r e^{-V(t)/3750} < 19,$$

where $V(t) = 100t$ hours. For this rule $\alpha' = .032$ and $\beta' = .051$.

Solution 2 (exact). The (B, A^*) test (2′) becomes

$$\tfrac{1}{19} < 3^r e^{-V(t)/3750} < 13.25.$$

For this rule $\alpha = \beta = .05$ exactly.

Problem 2. Compute $E_\theta(r)$ and $E_\theta(t)$ for $\theta = 0$, $\theta_1(= 2500)$, $s(= 4115)$, $\theta_0(= 7500)$, ∞.

Solution. For the (B, A) and (B, A^*) rules, respectively, the values of $E_\theta(r)$ are given in Remark 2 of Section 4. In the replacement case $E_\theta(t)$ is found most easily for all values of $\theta(\neq \infty)$ by using (12), $E_\theta(t) = (\theta/n)E_\theta(r)$.

$\theta =$	0	θ_1	s	θ_0	
$E_\theta(t) =$	0	175	333	227	using the (B, A) rule;
	0	155	297	220	using the (B, A^*) rule.

For $\theta = \infty$, the expected waiting time to reach a decision is given by t_∞, where $e^{-100t_\infty/3750} = 1/19$. This gives $t_\infty = E_\infty(t) = 110$.

Remark. More generally, in terms of B, n, θ_0, and k, we find

$$t_\infty = -\theta_0 \log B / n(k-1).$$

This means that if no items fail by t_∞, we stop experimentation at t_∞ with acceptance of H_0.

Problem 3. Assume that we are testing the hypothesis in problem 1 and that we are using the (B, A) rule. A sample of size 100 is placed on test. Items which fail are replaced by new items drawn from the same lot. The experiment is started at time $t = 0$. The first five failures occur at $x_1 = 20.1$ hours, $x_2 = 100.5$ hours, $x_3 = 121.7$ hours, $x_4 = 167.4$ hours, and $x_5 = 179.2$ hours, all times being measured from $t = 0$.

(a) Verify that no decision has been reached by time x_5.

(b) Verify that if the sixth failure has not yet occurred at 287.5 hours, measured from $t = 0$, we can stop experimentation at that time with acceptance of H_0.

Solution. It can be readily verified that, in this case, (5) becomes $-100 + 37.5r < t < 100 + 37.5r$. This region is drawn in Figures 1 and 2. The life test data are plotted by moving vertically so long as we are waiting for the next failure to occur and moving horizontally by one unit (in r) at each failure time. In Figure 1, the path crosses into the region of acceptance, when $r = 5$, at time $t = 100 + (37.5)5 = 287.5$. Since the sixth failure has not yet occurred, we can stop experimentation at $t = 287.5$ with acceptance of H_0.

Remark. As a matter of fact we happen to know in this case that the sixth failure occurs at $x_6 = 346.7$ hours. Thus, as indicated in Figure 1, we saved $346.7 - 287.5 = 59.2$ hours by observing the life test continuously in time.

Problem 4. The first seven failure times in a sample of 100 (with replacement) are $x_1 = 19.3$, $x_2 = 45.8$, $x_3 = 49.9$, $x_4 = 96.7$, $x_5 = 115.2$, $x_6 = 127.7$, and $x_7 = 131.2$. Verify that if the hypotheses being tested are those in Problem 1, then H_0 is rejected at time $x_7 = 131.2$ hours.

Solution. See Figure 2.

Remark. While the *acceptance* in Problem 3 is made *between* failure times x_5 and x_6, the *rejection* in Problem 4 is made *at* failure time x_7, with an excess over the boundary.

Problem 5. Find a truncated (nonsequential) replacement procedure for testing the hypothesis in Problem 1, using a constant sample size $n = 100$.

Solution. From results in [6], it can be verified that the truncated replacement procedure meeting the requirements is:

If $\min [x_{10}, 407.5] = 407.5$, truncate the experiment at 407.5 with acceptance of H_0.

If $\min [x_{10}, 407.5] = x_{10}$, truncate the experiment at x_{10} with acceptance of H_1.

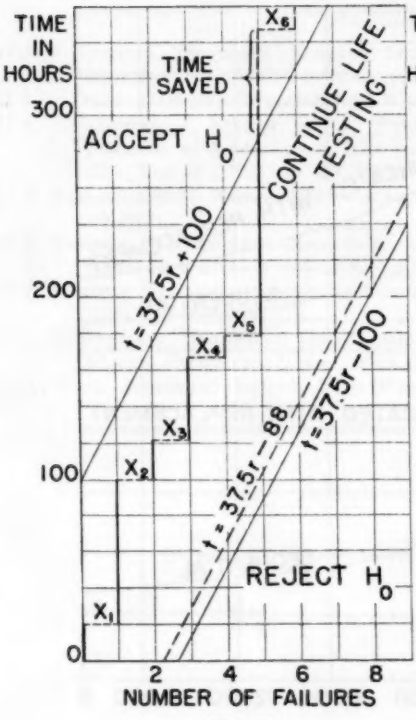

FIG. 1

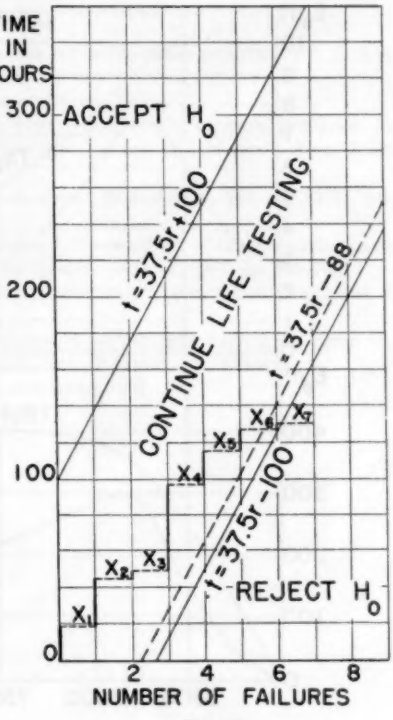

FIG. 2

The solid lines in Figures 1 *and* 2 *give the boundaries of a sequential with replacement test of* $H_0 : \theta_0 = 7500$ *hours against* $H_1 : \theta_1 = 2500$ *hours, with* $\alpha = \beta = .05$, *and* $n =$ *sample size* $= 100$, *when one uses a* (B, A) *rule. For the* (B, A^*) *rule the upper boundary remains the same, but the lower boundary becomes the dashed line,* $t = 37.5r - 88$. *Figure* 1 *gives a graphical treatment of Problem* 3. *Figure* 2 *gives a graphical treatment of Problems* 4 *and* 8.

The O.C. curves of this test procedure and of the one in Problem 1 are essentially the same.

Problem 6. Compute $E_\theta(r)$ and $E_\theta(t)$ for the plan in Problem 5 for $\theta = 0$, θ_1, s, θ_0, ∞.

Solution. From results in [6], $E_\theta(r) = 10$, 9.93, 8.75, 5.39, 0, and $E_\theta(t) = (\theta/n)E_\theta(r)$. For $\theta = 0$, θ_1, s, θ_0, ∞, respectively, $E_\theta(t) = 0$, 248, 360, 404.5, 407.5, respectively.

Remark. In Figure 3 we compare the $E_\theta(r)$ and $E_\theta(t)$ curves for Problems 2 (using a B, A^* rule) and 6. This will give some idea of the saving in the expected number of failures and time to reach a decision.

Problem 7. Find t_∞ in Problem 1 if $\alpha = \beta = .01$.

Solution. $t_\infty = -\theta_0 \log B / n(k - 1) = 230$. This is about twice the value of t_∞ when $\alpha = \beta = .05$.

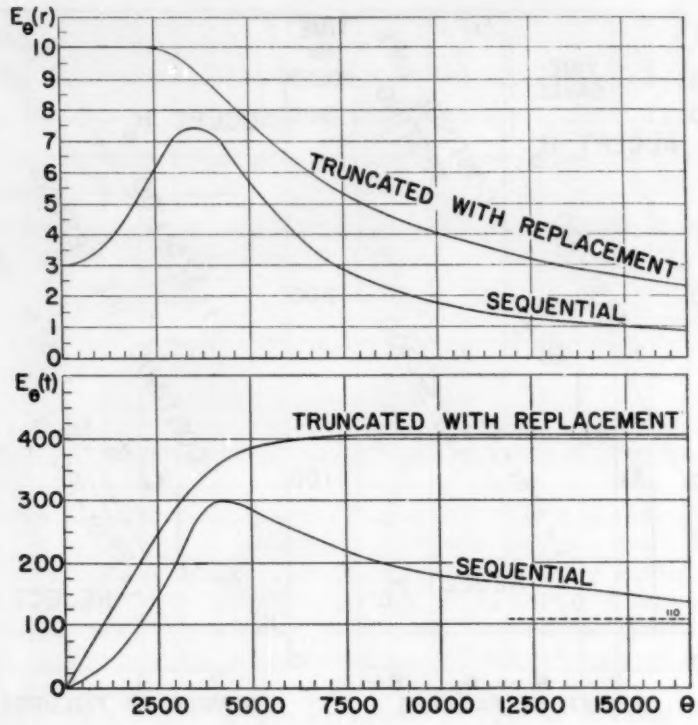

FIG. 3

Comparison of $E_\theta(r)$ (upper portion) and $E_\theta(t)$ (lower portion) curves for sequential and truncated with replacement plans. The O.C. curves for each plan are such that $L(\theta_0) = .95$ and $L(\theta_1) = .05$, with $\theta_0 = 7500$ and $\theta_1 = 2500$. The 110 dashed line gives the value which $E_\theta(t)$ approaches asymptotically as $\theta \to \infty$.

Problem 8. What happens in Problems 3 and 4 if a (B, A^*) rule is used?
Solution. The decision regions are

(B, A): $\quad -100 + 37.5r < t < 100 + 37.5r.$

$(B = 1/19, A^* = 13.25)$: $\quad -88 + 37.5r < t < 100 + 37.5r.$

No change occurs in the solution of Problem 3, since exactly the same decision boundary is being used for accepting H_0. However, the boundary used for rejecting H_0 when using (B, A^*) is shifted by .32 units (in r) to the left in Figures 1 and 2. For the data in Problem 4, this results in rejecting H_0 at time $x_6 = 127.7$ hours, since for $r = 6$, we have $-88 + 37.5r = 137$. This decision to reject H_0 is thus reached with one less item failed and 3.5 hours sooner than in Problem 4.

6. Acknowledgement. We wish to express our appreciation for the work of Mr. Hershel Harrison, who devoted many hours carrying out the calculations, the results of which are summarized in Remark 2 of Section 4.

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RANK SUM TESTS OF FIT

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Summary. This paper suggests several 'goodness of fit' test criteria, all having a linear form. The moment generating function and the limiting distribution of this linear form are obtained in Section 2. The best test criterion of this form for testing a simple hypothesis H_0 against a simple alternative hypothesis H_1 is shown, in Section 3, to be in general not independent of H_1.

The remainder of this paper deals with a special case of the linear form, that is, the rank sum test criterion. The distribution of this test criterion is derived in Section 4, its consistency is proved in Section 5, and some numerical asymptotic efficiencies are calculated in Section 6. Within a certain class of tests, the present test is shown, in Section 7, to be uniformly most powerful for a special family of alternatives.

1. Introduction. To test whether a sample $X_1, X_2, \cdots, X_m$ of a random variable X was obtained from a population having a completely specified continuous cdf $F(x)$, a number of tests (e.g., [8], [9]) have been based on a procedure by which the domain of the variable is divided into k sets of sizes so determined that the probability of each set under the null hypothesis is equal to $1/k$.

Evidently, for any given k, there are many possible ways of dividing the domain of the variable into sets with equal probabilities under the null hypothesis. For any given alternative hypothesis and given test procedure, one division might be better than another. On the other hand, for any given alternative and given division of the domain, one test might be better than another. We shall concern ourselves mainly with the latter problem. We assume that some knowledge of the alternative hypothesis is available.

The division used in this paper is as follows. Suppose (i) $f_0(x)$ and $f_1(x)$ are two completely specified continuous pdf's over a space R (which is either R_n or a subspace of R_n), and (ii) for every real number c the probability of the set $\{x;\ f_1(x)/f_0(x) = c\}$ is zero when the distribution is $f_0(x)$. Let H_i denote the hypothesis that the sample was obtained from $f_i(x)$, for $i = 0, 1$. Then R is to be divided into k disjoint sets $S_1, S_2, \cdots, S_k$ such that

$$S_j = \{x;\ c_{j-1} > f_1(x)/f_0(x) \geqq c_j\}, \qquad j = 1, 2, \cdots, k, \tag{1.1}$$

where the c_j, with $\infty = c_0 > c_1 > \cdots > c_k = 0$, are so determined that

$$p_{01} = p_{02} = \cdots = p_{0k} = 1/k, \tag{1.2}$$

$$p_{ij} = \int_{S_j} f_i(x)\, dx, \qquad j = 1, 2, \cdots, k;\ \ i = 0, 1. \tag{1.3}$$

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Denote by m_j the number of the sample values $X_1, X_2, \cdots, X_m$ falling in the set S_j for $j = 1, 2, \cdots, k$, and $\sum m_j = m$. Several tests based on a linear form of $m_1, m_2, \cdots, m_k$ will be proposed and their properties studied.

2. Distribution of a general linear form and its asymptotic normality. In this section we shall derive the moment generating function (mgf) of the linear form

$$L = \sum_{j=1}^{k} a_j m_j, \tag{2.1}$$

where $a_1, a_2, \cdots a_k$ are real constants (not all equal). We shall further show that its asymptotic distribution is normal.

THEOREM 1. *Under the hypothesis* H_i for $i = 0, 1$, *the* mgf *of* L *is given by*

$$M_i(t) = \left[\sum_{j=1}^{k} p_{ij} \exp(a_j t)\right]^m. \tag{2.2}$$

PROOF. Under hypothesis H_i, the distribution of $m_1, m_2, \cdots, m_k$ is known to be given by the multinomial distribution

$$w_i(m_1, m_2, \cdots, m_k) = m! \prod_{j=1}^{k} (p_{ij})^{m_j}/m_j!. \tag{2.3}$$

Hence, by application of the definition, the mgf of L in (2.1), under H_i, is given by $M_i(t)$ in (2.2).

THEOREM 2. *Under the hypothesis* H_i for $i = 0, 1$, *as* $m \to \infty$ *the distribution of* L *in* (2.1) *approaches the normal distribution with mean and variance*

$$\mu_i = m \sum_{j=1}^{k} a_j p_{ij}, \qquad \sigma_i^2 = m\left[\sum_{j=1}^{k} a_j^2 p_{ij} - \left(\sum_{j=1}^{k} a_j p_{ij}\right)^2\right]. \tag{2.4}$$

PROOF. Let $Y_1, Y_2, \cdots, Y_m$ be independent and identically distributed random variables such that $\Pr(Y_1 = a_j) = p_{ij}$ for $j = 1, 2, \cdots, k$ and $i = 0, 1$. Then L is distributed as $Y_1 + Y_2 + \cdots + Y_m$. Hence, by the central limit theorem, Theorem 2 is proved.

3. Most powerful test for testing H_0 against H_1. The main purpose of this section is to show that among the tests which depend only on $m_1, m_2, \cdots, m_k$, the most powerful test criterion for testing the simple null hypothesis H_0 against the simple alternative hypothesis H_1 is a linear function of $m_1, m_2, \cdots, m_k$, with coefficients a_j as functions of p_{1j} for $j = 1, 2, \cdots, k$.

THEOREM 3. *The best critical region for testing* H_0: $f(x) = f_0(x)$ *against* H_1: $f(x) = f_1(x)$ *is given by the subset of* $m_1, m_2, \cdots, m_k$:

$$\sum_{j=1}^{k} m_j \log(1/p_{1j}) \leqq c, \tag{3.1}$$

where c *is so determined that the level of significance is a preassigned size* α.

PROOF. By Neyman-Pearson's lemma and (2.3), it is easily seen that the best

critical region for testing H_0 against H_1 is given by those sets of $m_1, m_2, \cdots, m_k$ where

$$\frac{w_1(m_1, m_2, \cdots, m_k)}{w_0(m_1, m_2, \cdots, m_k)} = k^m \prod_{j=1}^{k} (p_{1j})^{m_j} \geqq c' \tag{3.2}$$

for some constant c'. Taking logarithms of both sides, we obtain the equivalent best critical region determined by (3.1).

It is seen from Theorem 3 that in order to find the best critical region, one has to find first the distribution of the linear function on the left side of (3.1). For large samples, the distribution can be approximated by the normal distribution, according to Theorem 2. For small samples, however, the determination of the distribution is involved.

4. Distribution of the rank sum criterion. In cases where the alternative hypothesis is not specified, the statistic on the left side of (3.1) is not applicable, since it involves p_{1j}'s. Furthermore, if the alternative hypothesis is stated in the composite form, no uniformly most powerful test can be found for most of the known distributions.

However, a class of alternatives may be such that they give rise to the same division of R (i.e., same sets $S_1, S_2, \cdots, S_k$). Then a rank sum test criterion as defined in (4.1) below seems to be a reasonable one to use, since in this case all the most powerful test criteria have the form (2.1) with the same property: $a_1 \leqq a_2 \leqq \cdots \leqq a_k$.

The word "rank" is here used in the sense that an observation falling in the set S_j is given the rank j, for $j = 1, 2, \cdots, k$. This is different from the usual usage of the word in two-sample problems. In the latter case, ranks of observations are determined by their relative positions, while here, ranks are determined by a given division of the space R.

THEOREM 4. *Let*

$$s = \sum_{j=1}^{k} j m_j. \tag{4.1}$$

Then, under H_0, *the* pdf *of* s *may be written as*

$$g(s; k, m) = k^{-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{s - 1 - rk}{m - 1}, \quad s = m, m + 1, \cdots, km. \tag{4.2}$$

Furthermore, $g(s; k, m)$ *is symmetrical,* that is,

$$g(m + v; k, m) = g(km - v; k, m), \quad v = 0, 1, \cdots, (k - 1)m. \tag{4.3}$$

PROOF. The pdf $g(s; k, m)$ is the coefficient of t^s in the power expansion of the distribution generating function

$$D(t; k, m) = k^{-m} \left(\sum_{j=1}^{k} t^j \right)^m. \tag{4.4}$$

This may be written as

$$D(t; k, m) = k^{-m}t^m(1 - t^k)^m(1 - t)^{-m}$$

$$= k^{-m} \sum_{p=0}^{m} \sum_{q=0}^{\infty} (-1)^{m-p} \binom{m}{p} \binom{m-1+q}{m-1} t^{m+q+k(m-p)}. \tag{4.5}$$

Letting $s = m + q + k(m - p)$, we obtain the pdf

$$g(s; k, m) = k^{-m} \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} \binom{s-1-k(m-p)}{m-1}. \tag{4.6}$$

Now, setting $r = m - p$, we get the pdf (4.2).

The symmetrical property (4.3) may be seen from the following argument. Since $g(m + v; k, m)$ is the coefficient of t^{m+v} in $D(t; k, m)$, it is the coefficient of $t^{-(m+v)}$ in $D(1/t; k, m)$, and hence the coefficient of t^{km-v} in $t^{(k+1)m} D(1/t; k, m)$. Thus (4.3) is proved, since

$$t^{(k+1)m}D(1/t; k, m) = D(t; k, m).$$

5. Consistency of the rank sum test. We shall now show that for any given k the rank sum test is consistent for a specified alternative or a class of alternatives.

THEOREM 5. *Let $k \geqq 2$ be a preassigned integer. Let the simple hypothesis H_0: $f(x) = f_0(x)$ be tested against the simple alternative hypothesis H_1: $f(x) = f_1(x)$, where $f_0(x)$ and $f_1(x)$ satisfy conditions* (i) *and* (ii) *in Section* 1.

Then, for any preassigned level of significance α, the power $u(f_1)$ of the rank sum test with index k approaches unity as the sample size m becomes indefinitely large.

PROOF. By Theorem 2, the statistic s defined in (4.1), under H_i for $i = 0, 1$, is asymptotically normally distributed. Let $N(x; \mu, \sigma)$ denote the cumulative normal distribution with mean μ and standard deviation σ. Then, for large m, the level of significance and the power of the test will be given approximately by

$$\alpha = N(\mu_0 - z_\alpha\sigma_0; \quad \mu_0, \sigma_0), \tag{5.1}$$

$$u(f_1) = N[(\mu_0 - \mu_1)/\sigma_1 - z_\alpha\sigma_0/\sigma_1; \quad 0, 1], \tag{5.2}$$

where z_α depends only on α, and μ_i and σ_i^2 for $i = 0, 1$, are given by (2.4) with $a_j = j$.

Since the ratio σ_0/σ_1 is constant, then to prove Theorem 5 it is sufficient to prove that $(\mu_0 - \mu_1)/\sigma_1$ increases as m increases, for any $k \geqq 2$, or equivalently,

$$\sum_{j=1}^{k} j(1/k - p_{1j}) > 0, \qquad k \geqq 2. \tag{5.3}$$

We shall first prove that, for any $k \geqq 2$,

$$p_{11} > 1/k, \qquad p_{1k} < 1/k. \tag{5.4}$$

In what follows, we shall denote by $S_j^{(k)}$ the sets defined in (1.1) and by $p_{ij}^{(k)}$ the quantities defined in (1.3), that is,

$$S_j^{(k)} = S_j, \qquad p_{ij}^{(k)} = p_{ij}, \qquad j = 1, 2, \cdots, k; i = 0, 1.$$

Since, by assumption, the probability of the set $\{x; f_1(x)/f_0(x) > 1\}$ is positive when the distribution is $f_1(x)$, then by (1.1), (1.2), and (1.3), we have $p_{11}^{(2)} > \frac{1}{2}$ and $p_{12}^{(2)} < \frac{1}{2}$. Thus, the inequalities (5.4) are true for $k = 2$.

For $k > 2$, we will prove (5.4) by considering the k sets $\{S_j^{(k)}\}$, for $j = 1, 2, \cdots, k$; the $2k$ sets $\{S_j^{(2k)}\}$ for $j = 1, 2, \cdots, 2k$; and the two sets $S_1^{(2)}$ and $S_2^{(2)}$. From (1.1), (1.2), and (1.3), it is easily seen that these sets satisfy the following relations

$$S_1^{(k)} = S_1^{(2k)} \cup S_2^{(2k)}, \qquad \bigcup_{j=1}^{k} S_j^{(2k)} = S_1^{(2)}. \tag{5.5}$$

Consequently, we find

$$p_{11}^{(k)} = p_{11}^{(2k)} + p_{11}^{(2k)}, \qquad \sum_{j=1}^{k} p_{1j}^{(2k)} = p_{11}^{(2)}. \tag{5.6}$$

Since $p_{11}^{(2k)} \geqq p_{12}^{(2k)} \geqq \cdots \geqq p_{1k}^{(2k)} \cdots$, we have

$$p_{11}^{(k)} = p_{11}^{(2k)} + p_{12}^{(2k)} \geqq \frac{2}{k} \sum_{j=1}^{k} p_{1j}^{(2k)} = \frac{2p_{11}^{(2)}}{k} > \frac{1}{k}. \tag{5.7}$$

Similarly, we can prove that $p_{1k}^{(k)} < 1/k$. Hence, the inequalities (5.4) are true for any $k \geqq 2$.

Now, it follows from a special case of the Tchebycheff's inequality (when $r = 1$ in Theorem 43 of [5]) that for any $k \geqq 2$, the inequality (5.3) is true, since $\{j\}$ and $\{1/k - p_{1j}^{(k)}\}$ are similarly ordered and $\sum(1/k - p_{1j}^{(k)}) = 0$. Thus, Theorem 5 is proved.

Theorem 5 implies that a one-sided rank sum test is consistent for the case where the null hypothesis H_0: $f(x) = f_0(x)$ is simple and the alternative hypothesis H_1^* consists of distributions $\{f_1^*(x)\}$ such that the ratios $\{r(x) = f_1^*(x)/f_0(x)\}$ are monotonic increasing (or decreasing) functions of x, provided that conditions (i) and (ii) in Section 1 are satisfied for each $f_1^*(x)$, when $f_1(x)$ is replaced by $f_1^*(x)$ there. On the other hand, a two-sided rank sum test also may be shown to be consistent under the following assumptions.

Suppose it is required to test the simple hypothesis H_0: $f(x) = f_0(x)$ against the composite hypothesis H_1^{**} which consists of two and only two classes C_1 and C_2 of those distributions such that the ratio $f_1^{**}(x)/f_0(x)$ is a monotonic decreasing function of x when $f_1^{**}(x)$ is in C_1 and a monotonic increasing function of x when $f_1^{**}(x)$ is in C_2. Suppose, also, that for each $f_1^{**}(x)$, conditions (i) and (ii) are satisfied, when $f_1(x)$ is replaced by $f_1^{**}(x)$. Then, we obtain

THEOREM 6. *Let $k \geqq 2$ be a preassigned integer, and α_1, α_2, and α be three preassigned positive numbers such that $\alpha_1 + \alpha_2 = \alpha$ where $0 < \alpha < 1$. Let h_1 and h_2, with $m \leqq h_1 < h_2 \leqq km$ be the values such that*

$$\sum_{s=m}^{h_1} g(s; k, m) = \alpha_1, \qquad \sum_{s=h_2}^{km} g(s; k, m) = \alpha_2. \tag{5.8}$$

Then, the test consisting of rejecting H_0 whenever $\sum jm_j \leqq h_1$ or $\sum jm_j \geqq h_2$ is consistent for testing H_0 against H_1^{**}.

PROOF. Theorem 6 is a corollary of Theorems 4 and 5, considering the two classes C_1 and C_2 separately.

Theorem 6 implies that the two-sided rank sum test is unbiased for testing H_0 against H_1^{**} for large samples. For small samples, it may or may not be unbiased.

6. Asymptotic efficiency of the rank sum test. In this section, we shall compare the rank sum test with one of the standard parametric tests, and obtain the power efficiency for large samples.

Suppose a sample is to be drawn from a population having a normal distribution with known variance σ^2 and unknown mean θ. Let the hypothesis H_0: $\theta = \theta_0$ be tested against the alternative hypothesis H_1: $\theta = \theta_1$, where $\theta_1 > \theta_0$. Further, let both the level of significance $\alpha = u(\theta_0)$ and the power of the test $u = u(\theta_1)$ be specified in advance, for example, $\alpha = .05$ and $u = .95$.

Let M and m be the sample sizes required by the likelihood ratio test and the rank sum test, respectively. We shall call $\mathcal{E} = M/m$ the *efficiency* of the rank sum test.

When $d = (\theta_1 - \theta_0)/\sigma$ is small, so that the sample size m is large, we can use the normal approximation (5.2) for the power function of the rank sum test. Hence, m is approximately given by

$$m = \left(\frac{z_\alpha \sqrt{(k^2 - 1)/12} + z_u \sqrt{\Sigma_1^k j^2 p_{1j} - (\Sigma_1^k j p_{1j})^2}}{\frac{1}{2}(k + 1) - \Sigma_1^k j p_{1j}} \right)^2, \tag{6.1}$$

while M is given by

$$M = (z_\alpha + z_u)^2 \sigma^2 / (\theta_1 - \theta_0)^2, \tag{6.2}$$

where z_α and z_u depend only on the specified values $\alpha = u(\theta_0)$ and $u = u(\theta_1)$, respectively. Consequently, the efficiency is given approximately by

$$\mathcal{E} = \frac{(z_\alpha + z_u)^2}{d^2} \left(\frac{\frac{1}{2}(k + 1) - \Sigma_1^k j p_{1j}}{z_\alpha \sqrt{(k^2 - 1)/12} + z_u \sqrt{\Sigma_1^k j^2 p_{1j} - (\Sigma_1^k j p_{1j})^2}} \right)^2. \tag{6.3}$$

The asymptotic efficiency may be found by evaluating the limit $\lim_{d \to 0} \mathcal{E}$. Thus, letting $n(x; \theta, \sigma)$ denote the normal density with mean θ and standard deviation σ, we obtain

THEOREM 7. *Let $k \geqq 2$ be a preassigned integer. Then, for any preassigned α and u with $u > \alpha$, the asymptotic efficiency (as $d \to 0$) of the rank sum test with index k is given by*

$$\mathcal{E}' = \frac{12}{k^2 - 1} \left[\sum_{j=1}^{k-1} n(y_j; 0, 1) \right]^2, \tag{6.4}$$

where the y_j, with $\infty = y_0 > y_1 > \cdots > y_k = -\infty$, are so determined that

$$N(y_{j-1}; 0, 1) - N(y_j; 0, 1) = 1/k, \qquad j = 1, 2, \cdots, k. \tag{6.5}$$

PROOF. It follows from the definition of p_{ij} that

$$p_{ij} = N(x_{j-1}; \theta_i, \sigma) - N(x_j; \theta_i, \sigma), \qquad j = 1, 2, \cdots, k; \quad i = 0, 1, \tag{6.6}$$

where the x_i, with $\infty = x_0 > x_1 > \cdots > x_k = -\infty$, are so determined that Equation (1.2) is satisfied. Since $\lim_{d\to 0} p_{1j} = p_{0j}$, we have

$$\mathcal{E}' = \lim_{d\to 0} \mathcal{E} = \frac{12}{k^2 - 1}\left[\lim_{d\to 0}\left(\frac{k+1}{2} - \sum_{j=1}^{k} j p_{1j}\right) \Big/ d\right]^2. \tag{6.7}$$

The limit in square brackets is an indeterminate form. However, the numerator and denominator satisfy the assumptions of l'Hospital's rule. Consequently, we obtain

$$\mathcal{E}' = \frac{12}{k^2 - 1}\left[\sum_{j=1}^{k} \sigma j\{n(x_{j-1}; \theta_0, \sigma) - n(x_j; \theta_0, \sigma)\}\right]^2. \tag{6.8}$$

From (6.5) and (6.6), it is easily seen that $y_j = (x_j - \theta_0)/\sigma$ for $j = 0, 1, \cdots, k$. Thus, the asymptotic efficiency $\mathcal{E}'$ may be written as

$$\begin{aligned}\mathcal{E}' &= \frac{12}{k^2 - 1}\left[\sum_{j=1}^{k} j\{n(y_{j-1}; 0, 1) - n(y_j; 0, 1)\}\right]^2 \\ &= \frac{12}{k^2 - 1}\left[\sum_{j=1}^{k} n(y_{j-1}; 0, 1) - k n(y_k; 0, 1)\right]^2.\end{aligned} \tag{6.9}$$

By assumption, $n(y_0; 0, 1) = n(y_k; 0, 1) = 0$. Therefore (6.9) becomes (6.4). This completes the proof of Theorem 7.

We remark that although the asymptotic efficiency $\mathcal{E}'$ was obtained under the assumption that α and u are preassigned, it is actually independent of them.

TABLE I

Power Efficiencies of the Rank Sum Tests

k \ d	0	.1	.2	.3	.4	.5
2	.637	.637	.636	.636	.635	.634
3	.793	.793	.793	.794	.794	.794
4	.856	.856	.857	.857	.858	.859
5	.888	.888	.889	.890	.891	.892
6	.906	.906	.906	.908	.909	.911
7	.918	.918	.919	.920	.921	.923
8	.926	.926	.927	.928	.929	.931
9	.931	.931	.932	.933	.934	.936
10	.935	.936	.937	.939	.940	.942

Table I gives the asymptotic efficiencies ($d = 0$) and certain approximate efficiencies ($d = .1, .2, .3, .4, .5$) for $k = 2, 3, \cdots, 10$. The approximate efficiencies were obtained under the assumption that $\alpha = 1 - u$ (where α is small).

For $0 < d \leqq .5$, the approximate efficiencies in Table I are very close to the exact ones. We note also that $.637 = 2/\pi$ is just the power efficiency of the large sample binomial test, since the rank sum test reduces to the binomial test when $k = 2$ (see also [2], [4]).

7. Power of a uniformly most powerful rank sum test. We have shown that the rank sum tests are consistent, and the numerical calculations show that their asymptotic efficiencies are approximately over 90% for the normal alternatives, when $k \geqq 6$. However, these are only the limiting behavior of the tests. Naturally, the power of the rank sum tests for small samples is also desirable. It is the purpose of this section to investigate the power of the rank sum tests for a special family of alternatives. This family is so chosen that among the tests which depend only on $m_1, m_2, \cdots, m_k$, the rank sum test is uniformly most powerful for testing the uniform distribution against such a family.

Let a family of cdf's be given by

$$F(x; A) = \begin{cases} x, & A = 1, \\ (A^x - 1)/(A - 1), & A > 1, \end{cases} \qquad 0 \leqq x \leqq 1. \tag{7.1}$$

Let the hypothesis H_0': $A = 1$ be tested against the alternative hypothesis H_1': $A > 1$. Then,

THEOREM 8. *Among the tests which depend only on $m_1, m_2, \cdots, m_k$, the rank sum tests are uniformly most powerful for testing H_0' against H_1'. The power functions are given by*

$$u(A; k, m) = \frac{k^m A^m (A^{1/k} - 1)^m}{(A - 1)^m} \sum_{s=m}^{c^*} g(s; k, m) A^{-s/k}, \tag{7.2}$$

where the level of significance is

$$\sum_{s=m}^{c^*} g(s; k, m) = \alpha. \tag{7.3}$$

PROOF. Under H_0', $F(x; A)$ is just the cdf for the uniform distribution on $(0, 1)$. Therefore, according to the assumptions in Section 1, we have

$$S_j = \left(\frac{k-j}{k}, \frac{k-j+1}{k}\right), \qquad j = 1, 2, \cdots, k. \tag{7.4}$$

Hence

$$p_{1j} = \frac{A^{(k-j+1)/k} - A^{(k-j)/k}}{A - 1} = \frac{A^{-j/k}(A^{(k+1)/k} - A)}{A - 1}, \qquad j = 1, 2, \cdots, k. \tag{7.5}$$

Thus, by (3.1), the best critical region is given by

$$\sum_{j=1}^{k} m_j \log\left(\frac{1}{p_{1j}}\right) = \log A^{1/k} \sum_{j=1}^{k} j m_j + m\left[\log (A - 1) - \log (A^{(k+1)/k} - A)\right] \leqq c. \tag{7.6}$$

Since $A > 1$ and the second term is constant, we have the equivalent best critical region determined by

$$\sum_{j=1}^{k} j m_j \leqq c^*. \tag{7.7}$$

It is obvious that for any given level of significance α the best critical region determined by (7.7) is independent of the parameter A. Consequently, the rank sum test is uniformly most powerful for all $A > 1$. Thus, we have proved the first assertion of Theorem 8.

The power function of the test is derived as follows:

$$
\begin{aligned}
u(A; k, m) &= \sum m! \prod_{j=1}^{k} \frac{(p_{1j})^{m_j}}{m_j!} \\
&= \sum m! \prod_{j=1}^{k} \frac{A^{-jm_j/k}(A^{(k+1)/k} - A)^{m_j}}{(A-1)^{m_j} m_j!} \\
&= \frac{(A^{(k+1)/k} - A)^m}{(A-1)^m} \sum m! \prod_{j=1}^{k} \frac{A^{-jm_j/k}}{m_j!} \qquad (7.8) \\
&= \frac{(A^{(k+1)/k} - A)^m k^m}{(A-1)^m} \sum m! \prod_{j=1}^{k} \frac{A^{-jm_j/k}}{k^m m_j!} \\
&= \frac{A^m (A^{1/k} - 1)^m k^m}{(A-1)^m} \sum_{s=m}^{c^*} g(s; k, m) A^{-s/k},
\end{aligned}
$$

where the summation $\sum$ is extended over all possible combinations of $m_1, m_2, \cdots, m_k$ such that $m \leqq \sum jm_j \leqq c^*$ and $\sum m_j = m$. This completes the proof of Theorem 8.

In Figure 1 are plotted six power curves $u(A; k, m)$ for $k = 2, 3, 6$ and $m = 4, 6$, comparing the power curves of the rank sum tests with indices $k = 3, 6$ with that of the binomial tests; when $k = 2$, the rank sum test reduces to the binomial test. The levels of significance (nonrandomized) are

$u(1; 2, 4) = .0625$, $u(1; 3, 4) = .0617$, $u(1; 6, 4) = .0540$,
$u(1; 2, 6) = .1094$, $u(1; 3, 6) = .1070$, $u(1; 6, 6) = .0965$.

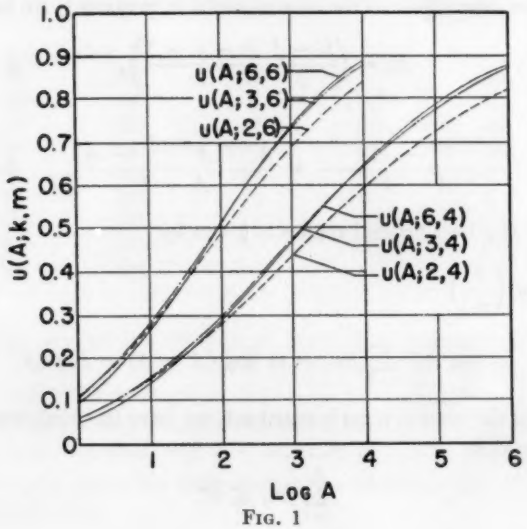

FIG. 1

The family of distributions $F(x;\ A)$ not only yields nice power functions for the rank sum tests, but also possesses the nice property that A plays the same role as a location parameter in the sense that the mean $\mu = A/(A-1) - 1/\log A$ is a monotonic increasing function of A. For, letting $y = \log A$, we have

$$\mu = e^y/(e^y - 1) - 1/y, \tag{7.9}$$

and hence

$$\frac{d\mu}{dy} = \frac{(e^y-1)^2 - y^2e^y}{y^2(e^y-1)^2} = \frac{(e^y - 1 + ye^{y/2})(e^y - 1 - ye^{y/2})}{y^2(e^y-1)^2}. \tag{7.10}$$

Now, when $y > 0$, we have

$$e^y - 1 - ye^{y/2} = \left(\frac{1}{3!} - \frac{1}{2^2 2!}\right)y^3 + \left(\frac{1}{4!} - \frac{1}{2^3 3!}\right)y^4 + \cdots > 0. \tag{7.11}$$

Consequently, μ is a monotonic increasing function of y, for $y > 0$, and hence of A, for $A > 1$.

Since A plays the role of a location parameter, the class of alternatives $F(x; A)$ may be regarded as representative of the principal types of deviation from the null hypothesis. To show the general shape we plot, in Figure 2, five curves of this family along with five curves of normal alternatives $N(x;\ \theta,\ \sigma)$, where the normal distributions are transformed by $\xi = N(x;\ \theta_0,\ \sigma)$. The family $F(x;\ A)$ is shown by solid curves, while the family $N(x;\ \theta,\ \sigma)$ is shown by dotted curves. In the latter case, the letter d represents the distance of the mean θ from the hypothetical mean θ_0 in terms of the standard deviation σ, that is, $d = (\theta - \theta_0)/\sigma$.

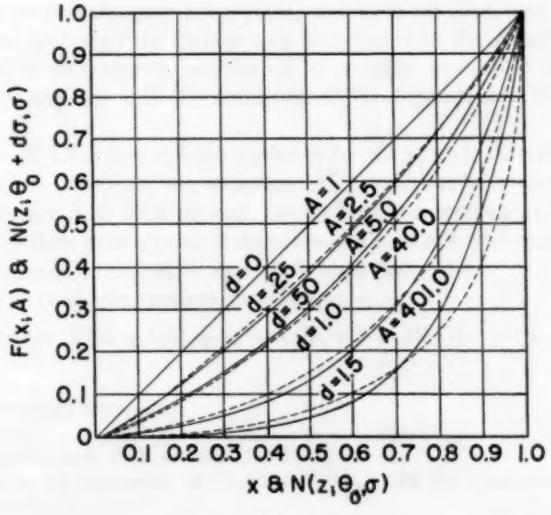

FIG. 2

Figure 2 shows that certain normal alternatives (when d is small) can be approximated by certain alternatives $F(x; A)$. Therefore, for a fixed level of significance their corresponding power curves can be used as approximations to each other.

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RANDOM FUNCTIONS SATISFYING CERTAIN LINEAR RELATIONS, II[1].

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0. Introduction and summary. We consider here another aspect of a problem mentioned in a previous paper [2]. We shall be concerned with one-dimensional, real-valued random functions (r.f.) $X(t)$, defined for all t and such that any sample taken at equidistant t-points satisfies a linear relation which is an analogue of one or other of the stochastic difference relations which are used for the analysis of discrete-parameter time-series. More specifically, we assume that there exist k continuous and real-valued functions $\alpha_1(h), \cdots, \alpha_k(h)$ of $h \geqq 0$ such that for any $h > 0$ and any t, the sequence

$$\{X(t + [n + k]h) + \alpha_1(h)X(t + [n + k - 1]h) + \cdots + \alpha_k(h)X(t + nh)\}, \quad n = 0, \pm 1, \cdots, \tag{1}$$

satisfies certain conditions about independence or noncorrelation. In Section 1, we consider hypotheses concerning correlation, and find that the functions $\alpha_j(h)$ are restricted to certain forms. We also find that the assumption of zero serial correlations in the sequence (1) for all $h > 0$ implies that $X(t)$ is deterministic. In Section 2, we consider hypotheses concerning independence, and find the functions $\alpha_j(h)$ to be restricted as before.

1. Hypotheses about correlation. In this section, we assume $X(t)$ to have a finite variance and to be continuous in mean square for all t. We shall consider first processes with stationary covariance, and then all processes with continuous covariance. Although the former case is included in the latter, we find that the restriction of stationarity enables us to consider somewhat less restrictive hypotheses. By analogy with the definition of "m-dependence" [3], we shall use the following

Definition. If $\{X_n\}$ has a finite variance for all n, and X_n and $X_{n'}$ are noncorrelated for $|n - n'| > m$, the sequence $\{X_n\}$ is said to be *m-correlated.*

Theorem 1. *Let $X(t)$ be a random function with a continuous, stationary covariance, and let there exist a positive integer m and a set of k real-valued, continuous functions $\alpha_1'(h), \alpha_2'(h), \cdots, \alpha_k'(h)$ of $h \geqq 0$, such that*

(i) *for any $h > 0$, the random variables*

$$Y(\alpha'; h; n) = X([n + k]h) + \alpha_1'(h)X([n + k - 1]h) + \cdots + \alpha_k'(h)X(nh), \quad n = 0, \pm 1, \cdots, \tag{2}$$

form an m-correlated sequence;

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(ii) *there is no* $h > 0$ *for which such a relation holds for some* m *with less than* k *coefficients, say* $\beta_1(h), \beta_2(h), \cdots, \beta_l(h)$, *with* $l < k$, *instead of the* $\alpha_j'(h)$.

Then there exists a uniquely determined set of k *continuous, real-valued functions* $\alpha_1(h), \cdots, \alpha_k(h)$ *with the following properties:*

(a) *For any* $h > 0$, *the sequence* $\{Y(\alpha; h; n)\}$, for $n = 0, \pm 1, \cdots$ *is* $(k-1)$*-correlated, that is to say* $m = k - 1$. *It is* $(k-2)$*-correlated if and only if* $X(t)$ *is deterministic and satisfies the relation*

$$X(t + kh) + \alpha_1(h)X(t + [k-1]h) + \cdots + \alpha_k(h)X(t) = 0 \tag{3}$$

with probability 1 *for any* $h > 0$ *and any* t.

(b) *Every root* $\gamma_j(h)$ *of the equation in* x,

$$x^k + \alpha_1(h)x^{k-1} + \cdots + \alpha_k(h) = 0, \tag{4}$$

can be written in the exponential form $\exp(\lambda_j h)$, *where the* λ_j *are independent of* h *and are roots, of respective multiplicities* k_j, *of an algebraic equation of degree* k *with real coefficients which are independent of* h. *Further, the real parts of the* λ_j *are nonpositive, and* $k_j = 1$ *if* λ_j *is purely imaginary. The root* $\exp(\lambda_j h)$ *of* (4) *has multiplicity* k_j.

PROOF. It is no restriction to assume $E\{X(t)\} = 0$ and $E\{X^2(t)\} = 1$. We shall also write $\rho(h)$ for $E\{X(t)X(t+h)\}$ and define the operator U_h by $U_h f(t) = f(t+h)$.

Taking $\alpha_0'(h) = 1$, and using the m-correlation property of the sequence (2), we have for any $h > 0$ and for any integer $n > \max(m - k, 0)$

$$\left\{\sum_{j=0}^{k} \alpha_j'(h)U_h^{k-j}\right\}\left\{\sum_{j=0}^{k} \alpha_j'(h)U_h^{j}\right\}\rho(nh) = 0. \tag{5}$$

Thus for any $h > 0$, we have a linear difference equation satisfied by the sequence $\rho(nh)$. Hence, $\rho(nh)$ can be expressed as a function of n and the roots of the characteristic equation

$$\left\{\sum_{j=0}^{k} \alpha_j'(h)x^{k-j}\right\}\left\{\sum_{j=0}^{k} \alpha_j'(h)x^{j}\right\} = 0. \tag{6}$$

For the present, let us consider some fixed $h > 0$, and let $\gamma_j(h)$ for $j = 1, 2, \cdots, p, p+1, \cdots, p+p'$, be all the distinct roots, of moduli not exceeding 1, of equation (6); let k_j be the multiplicity of the root $\gamma_j(h)$, and let

$$|\gamma_j(h)| \begin{cases} = 1 & \text{for } j = 1, 2, \cdots, p, \\ < 1 & \text{for } j = p+1, p+2, \cdots, p+p'. \end{cases} \tag{7}$$

Since the reciprocal of any root of (6) is also a root, we see that the whole set of distinct roots consists of

$$\gamma_j(h), \quad j = 1, 2, \cdots, p+p', \qquad 1/\gamma_j(h), \quad j = p+1, \cdots, p+p',$$

each root with suffix j being of multiplicity k_j. Hence, for all integers $n > \max(m - k, 0)$, we get from (5)

$$\rho(nh) = \sum_{j=1}^{p} \gamma_j^n(h)\left\{\sum_{s=0}^{k_j-1} a_{js}(h)n^s\right\} + \sum_{j=p+1}^{p+p'} \gamma_j^n(h)\left\{\sum_{s=0}^{k_j-1} b_{js}(h)n^s\right\} + \sum_{j=p+1}^{p+p'} \gamma_j^{-n}(h)\left\{\sum_{s=0}^{k_j-1} c_{js}(h)n^s\right\}, \quad n > \max(m - k, 0), \tag{8}$$

where the a's, b's and c's are constant with respect to n, but continuous functions of h. Since $|\rho(nh)| \leqq 1$, the last term in (8) drops out entirely and there can be no positive powers of n in the first term on the right side of (8), so that

$$\rho(nh) = \sum_{j=1}^{p} \gamma_j^n(h)a_{j0}(h) + \sum_{j=p+1}^{p+p'} \gamma_j^n\left\{\sum_{s=0}^{k_j-1} b_{js}(h)n^s\right\}, \quad n > \max(m - k, 0). \tag{9}$$

Therefore, the sequence $\{\rho(nh)\}$ satisfies a homogeneous, linear difference equation (with right member zero) of order $k' = p + \sum_{j=1}^{p'} k_{p+j}$. Now, the roots of (6) comprise the roots of

$$x^k + \alpha_1'(h)x^{k-1} + \cdots + \alpha_k'(h) = 0 \tag{10}$$

and their reciprocals. Hence, it follows that every root of (6) of unit modulus is of even multiplicity. Consequently, it is easy to see that $k' \not> k$. On the other hand, if $k' < k$, we can use the fact that the sequence $\{\rho(nh)\}$ satisfies a linear relation of order k' to show that condition (ii) of the theorem is violated. Thus $k' = k$, and for every $h > 0$ there exist k real numbers $\alpha_1(h), \cdots, \alpha_k(h)$ such that.

$$\sum_{j=0}^{k} \alpha_j(h)\rho([n - j]h) = 0 \quad \text{any } n > m_0 = \max(m, k) \tag{11}$$

This set of k numbers is unique for every $h > 0$, since if there were two non-identical relations of order k of the type of (11), we could derive from them a relation of lower order, and this would contradict assumption (ii) of the theorem. The equation

$$x^k + \alpha_1(h)x^{k-1} + \cdots + \alpha_k(h) = 0 \tag{12}$$

has p distinct, nonrepeated roots $\gamma_j(h)$, where $j = 1, 2, \cdots, p$, of unit modulus, and p' distinct roots $\gamma_j(h)$, where $j = p + 1, \cdots, p + p'$, of moduli less than 1 and respective multiplicities k_j.

From (11) we know that for any $h > 0$ and any integer $r > 0$, the sequence $\rho(nh/r)$, for $n = m_0 + 1, \quad m_0 + 2, \cdots$, satisfies the relation

$$\sum_{j=0}^{k} \alpha_j(h/r)\rho\{(n - j)h/r\} = 0. \tag{13}$$

From (13) we shall now derive a relation satisfied by the sequence $\rho(nh)$. For this purpose, let $\alpha_1(r; h/r), \alpha_2(r; h/r), \cdots, \alpha_k(r; h/r)$ be the set of numbers determined uniquely by the property that the roots of

$$x^k + \alpha_1(r; h/r)x^{k-1} + \cdots + \alpha_k(r; h/r) = 0 \tag{14}$$

are the rth powers $\gamma_j^r(h/r)$ of the characteristic roots of (13). Let β_i, for $i = 0, 1, 2, \cdots$, be defined by the identity

$$\sum_{i=0}^{rk-k} \beta_i x^{rk-k-i} \equiv \frac{x^{kr} + \alpha_1(r; h/r)x^{(k-1)r} + \cdots + \alpha_k(r; h/r)}{x^k + \alpha_1(h/r)x^{k-1} + \cdots + \alpha_k(h/r)}. \tag{15}$$

Then it is easily verified that

$$\sum_{i=0}^{rk-k} \beta_i \sum_{j=0}^{k} \alpha_j(h/r)\rho\{(nr - i - j)h/r\} = \sum_{j=0}^{k} \alpha_j(r; h/r)\rho\{(n - j)rh/r\}. \tag{16}$$

Consequently,

$$\sum_{j=0}^{k} \alpha_j(r; h/r)\rho\{(n - j)h\} = 0 \qquad n > m_0. \tag{17}$$

But the uniqueness of (11) gives us $\alpha_j(r; h/r) = \alpha_j(h)$ and hence $\gamma_j^r(h/r) = \gamma_j(h)$. Then, by a standard argument using continuity, we have $\gamma_j(h) = \exp(\lambda_j h)$ for all $h \geqq 0$, where λ_j is some constant. We have thus proved (b).

Next, from (9) by equating the expressions for $\rho(nh)$ and $\rho(h')$, so that $h' = nh$, we get

$$a_{j0}(h) = a_{j0}(nh), \qquad b_{js}(h)n^s = b_{js}(nh).$$

This gives us $a_{j0}(h) = a_{j0}$ and $b_{js}(h) = b_{js}h^s$, where the a_{j0} and b_{js} are constants. Hence,

$$\rho(h) = \sum_{j=1}^{p'} a_{j0} \exp(\lambda_j h) + \sum_{j=p+1}^{p+p'} \sum_{s=0}^{kj-1} b_{js} h^s \exp(\lambda_j h), \qquad h > 0. \tag{18}$$

From this form of $\rho(h)$, we immediately have

$$E\{Y(\alpha; h; 0)X(t)\} = 0 \qquad h \geqq 0, \quad t \leqq 0. \tag{19}$$

It follows trivially that the Y-sequence is $(k - 1)$-correlated. Thus the first part of (a) is proved.

Finally, suppose that the Y-sequence is $(k - r)$-correlated for some $r \geqq 2$. This together with (19) implies that

$$E\{Y(\alpha; h; j)X(rh)\} = 0 \qquad j = 1, 2, \cdots. \tag{20}$$

Also, since $(k - r)$-correlation with $r \geqq 2$ implies $(k - 2)$-correlation, we have

$$E\{Y(\alpha; rh; 0)X(rh)\} = 0. \tag{21}$$

Now, by the procedure used in (14) to (16), we can obtain the relation

$$Y(\alpha; rh; 0) = \beta_0 Y(\alpha; h; rk - k) + \cdots + \beta_{rk-k} Y(\alpha; h; 0),$$

which leads to

$$E\{Y(\alpha; h; 0)X(rh)\} = 0. \tag{22}$$

This implies that the Y-sequence is $(k - r - 1)$-correlated. Hence, by induction we can show that the assumption of $(k - 2)$-correlation in the Y-sequence implies

$$E\{Y^2(\alpha; h; 0)\} = 0, \tag{23}$$

which proves the second part of (a).

REMARKS. The uniqueness of the coefficients $\alpha_j(h)$ is a consequence of the restriction $|\gamma_j(h)| \leqq 1$. There is another unique set of k coefficients under the restriction $|\gamma_j(h)| \geqq 1$; in this case the roots are reciprocals of those in the previous. This is merely a consequence of the fact that, on account of stationarity, the covariance function of $Y(t) = X(-t)$ is the same as that of $X(t)$; the reversal of the t-axis transforms the linear function with characteristic roots $\gamma_j(h)$ into one whose roots are $1/\gamma_j(h)$.

This result may be compared with some of the results of Doob [1]. The first $k-1$ derivatives in mean square of a stationary r.f. exist if and only if $\rho(h)$ has a derivative of order $2(k-1)$ at the origin, and this condition is completely equivalent to the finiteness of the moment of order $2(k-1)$ of the spectral function. From these facts and Theorems 3.10, 4.8 and 4.9 of Doob [1], we can derive the

COROLLARY. *A one-dimensional, stationary Gaussian process is a component of a k-dimensional* "t.h.G.M." *process if and only if it is a Gaussian process satisfying the assumptions of Theorem* 1. *Furthermore, it is a* "t.h.G.M_k" *process if and only if, in addition, its $(k-1)$th derivative in mean square exists.*

We shall now state a result similar to Theorem 1 which holds without the restriction of stationarity.

THEOREM 2. *Let $X(t)$ be continuous in mean square for all t, with $E\{X(t)\} = 0$. Let there exist a positive integer m, and k continuous functions, $\alpha_1(h), \cdots, \alpha_k(h)$, of $h \geqq 0$, such that*

$$E\{[X(t+kh) + \alpha_1(h)X(t+[k-1]h) + \cdots + \alpha_k(h)X(t)]X(t-nh)\} = 0, \qquad n = m,\ m+1, \cdots, \tag{24}$$

assuming that such a relation does not hold with any other set of α's for any h (and any m).

Then the roots of (4) *can be written in the form* $\exp(\lambda_j h)$, *where the λ_j are the roots of an algebraic equation of degree k with real coefficients. Furthermore,* (24) *is true for all integers $n \geqq 0$, and if it also holds for $n = -1$, then it holds for all n, and Equation* (3) *is true with probability* 1.

The proof is similar to that of Theorem 1, and is thus omitted.

2. Hypotheses of independence. Finally, we shall consider a result of this type, but with noncorrelation replaced by independence. We shall see that the Y-sequence is necessarily $(k-1)$-dependent, but we are unable to determine whether it can be $(k-2)$-dependent without triviality.

THEOREM 3. *Let $X(t)$ be a real-valued* r.f., *continuous in probability for all t. Let there exist a positive integer m, and k real-valued, continuous functions $\alpha_1(h), \cdots, \alpha_k(h)$, of $h \geqq 0$ such that*

(i) *for any t and any $h > 0$, the random variables*

$$Y(t, h; n) = X(t + [n+k]h) + \alpha_1(h)X(t + [n+k-1]h) + \cdots + \alpha_k(h)X(t+nh), \qquad n = 0, \pm 1, \cdots,$$

form an m-dependent sequence;

(ii) *there is no other set of* α's *for which this is true for some* m; *and*

(iii) $\alpha_k(h) \neq 0$ *for any* h.

Then the roots of (4) *can be written in the form* $\exp(\lambda_j h)$, *where the* λ_j *are roots of an algebraic equation of degree* k *with real coefficients. Furthermore,* $m = k - 1$.

PROOF. As before, we take any positive h and any positive integer $r > m - k$, and let $h' = h/r$. On account of m-dependence, we know that, for any integer $l > m$ and any positive integers p and q, the set of random variables $\{Y(t, h'; n)\}$ for $n = 0, 1, \cdots, (p + k)r$ is independent of the set $\{Y(t, h'; -l - n)\}$ for $n = 0, 1, \cdots, (q + k)r$. Now let

$$S_n(r) = \sum_{j=0}^{kr-k} \beta_j Y(t, h'; \quad nr + kr - k - j), \qquad n = 0, 1, \cdots, p. \tag{25}$$

$$\begin{aligned} T_n(r, l) &= \sum_{j=0}^{kr-k} \beta_j Y(t, h'; \quad -nr - l - j) \\ &= \sum_{j=0}^{kr-k} \beta_j Y[t + (k - l)h', h'; \quad -nr - k - j], \end{aligned} \qquad j = 0, 1, \cdots, q, \tag{26}$$

where the β_j are as in (15). Then the random variables $S_n(r)$ for $n = 0, 1, \cdots, p$ are independent of $T_n(r, l)$ for $n = 0, 1, \cdots, q$.

By the method used in (14) to (16), we shall now derive a new linear difference function which generates a sequence having the properties of $\{Y(t, h; \quad n)\}$. For this purpose, let

$$\begin{aligned} Y'(t, h, n) &= X[t + (n + k)h] \\ &\quad + \alpha_1(r; h')X[t + (n + k - 1)h] + \cdots + \alpha_k(r; h')X[t + nh]. \end{aligned} \tag{27}$$

Then it is easily verified that

$$S_n(r) = Y'(t, h; \quad n), \qquad T_n(r, l) = Y'[t + (k - l)h', h; \quad -n - k]. \tag{28}$$

For $l = r + k$, we have from the second expression that

$$\begin{aligned} T_n(r, r + k) &= Y'(t - rh', h; -n - k) \\ &= Y'(t, h; -n - k - 1). \end{aligned} \tag{29}$$

Since the $S_n(r)$ for $n = 0, 1, \cdots, p$ are independent of $T_n(r, \quad r + k)$ for $n = 0, 1, \cdots, q$, it is clear that the Y'-sequence is k-dependent. Hence, by assumption (ii) of the theorem

$$Y'(t, h; n) = Y(t, h; n), \tag{30}$$

so that $\alpha_j(r; h/r) = \alpha_j(h)$. This leads as before to $\gamma_j(h) = \exp(\lambda_j h)$.

Finally, from (28) and (30) we know that for any $l > m$ and any positive integers p and q, the random variables $Y(t, h; n)$ for $n = 0, 1, \cdots, p$ are independent of $Y[t + (k - l)h/r, \quad h; \quad -n - k]$ for $n = 0, 1, \cdots, q$. With $l = m + 1$, as $r \to \infty$ the second set of random variables converges in probability to $Y[t, h; -n - k]$ for $n = 0, 1, \cdots, q$ and therefore is independent of $Y(r, h; n)$ for $n = 0, 1, \cdots, p$. In other words, the Y-sequence is $(k - 1)$-dependent, which is to say that $m = k - 1$.

COROLLARY. *Any random function $X(t)$ having the property that for any t and any $h > 0$ the increments $[X(t+h) - X(t)]$, $[X(t+2h) - X(t+h)]$, $\cdots$ form an m-dependent sequence, where m is some nonnegative integer, is a random function with independent increments ("additive process").*

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ON THE DISTRIBUTION OF THE SAMPLE MEDIAN[1]

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1. Summary. Upper and lower bounds are obtained for the cumulative distribution function of the sample median of a sample of size $2n + 1$ drawn from a continuous population. It is shown that if the parent population is normal, then the distribution of the sample median tends "rapidly" to normality. Other kinds of parent populations are also discussed.

2. Introduction. Let a continuous population be given with cdf $F(x)$ (cumulative distribution function) and median ξ (assumed to exist uniquely). For a sample of size $2n + 1$, let $\tilde{x}$ denote the sample median. The distribution of $\tilde{x}$, under certain conditions, is known ([2], p. 369) to be asymptotically normal with mean ξ and variance $\sigma_n^2 = 1/4\,[f(\xi)]^2(2n + 1)$, where $f(x) = F'(x)$ is the pdf (probability density function).

Several authors, among them Hojo [4] and Cadwell [1], have stated that numerical investigations showed that, if the parent population is normal, "the convergence (of the distribution of $\tilde{x}$) to normality is surprisingly fast." However, no mathematical proof or disproof seems ever to have been given for this experimental result.

This paper shows mathematically that the findings are correct. Upper and lower bounds are obtained for $P[-x < (\tilde{x} - \xi) / \sigma_n < y]$ in (8) and (9). If no very high accuracy is required, these bounds are reduced to a simpler form in (10) and (11). Examination of these bounds makes it evident that the distribution of $(\tilde{x} - \xi) / \sigma_n$ tends "rapidly" to normality.

Rectangular and Laplace parent populations are briefly discussed. It seems that in these cases the distribution of $(\tilde{x} - \xi) / \sigma_n$ tends to normality at a "much slower speed."

3. Upper and lower bounds. Let $F(x)$ and $f(x)$ be respectively the cdf and pdf of a certain population whose median is ξ. If $g(x)$ is the pdf of the sample median $\tilde{x}$ of a sample of size $2n + 1$, then

$$g(x) = C_n[F(x)]^n[1 - F(x)]^n f(x), \qquad C_n = (2n + 1)! / n!n!.$$

If $f(\xi) \neq 0$ and $f'(x)$ is continuous in some neighborhood of $x = \xi$, then $\tilde{x}$ is known ([2], p. 369) to have an asymptotically normal distribution with mean ξ and variance

$$\sigma_n^2 = \frac{1}{4[f(\xi)]^2\,(2n + 1)}.$$

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For finite n, let the cdf of $(\tilde{x} - \xi) / \sigma_n$ be $H(x)$. Then for any $y > 0$,

$$H(y) - \tfrac{1}{2} = \int_{\xi}^{\xi + y\sigma_n} g(t)\, dt = C_n \int_{1/2}^{F(\xi + y\sigma_n)} u^n (1 - u)^n\, du$$

$$= (\tfrac{1}{2})^{2n} C_n \int_0^{F(\xi + y\sigma_n) - 1/2} (1 - 4v^2)^n\, dv.$$

Applying the transformations

$$v = \begin{cases} av', & 0 < a \leqq 1; \\ bv', & 1 \leqq b; \end{cases} \qquad v' = \tfrac{1}{2}\sqrt{1 - \exp\left[-t^2/(2n+1)\right]},$$

we obtain without difficulty

$$(1) \quad H(y) - \tfrac{1}{2} \geqq aB_n \int_0^{t_1} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{n+1}{2n+1}\, t^2\right] h_1\left(\frac{t}{\sqrt{2n+1}}\right) dt, \qquad 0 < a \leqq 1,$$

$$(2) \quad H(y) - \tfrac{1}{2} \leqq bB_n \int_0^{t_2} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{n}{2n+1}\, t^2\right] h_2\left(\frac{t}{\sqrt{2n+1}}\right) dt, \qquad b \geqq 1,$$

where $B_n = (\tfrac{1}{2})^{2n+1} C_n \sqrt{2\pi} / \sqrt{2n+1}$ and

$$(3) \quad h_1(t) = t / \sqrt{1 - \exp(-t^2)}, \qquad h_2(t) = t \exp(-t^2) / \sqrt{1 - \exp(-t^2)};$$

$$(4) \quad \begin{aligned} t_1 &= \sqrt{-(2n+1) \log\left\{1 - (4/a^2)\left[F(\xi + y\sigma_n) - \tfrac{1}{2}\right]^2\right\}}, \\ t_2 &= \sqrt{-(2n+1) \log\left\{1 - (4/b^2)\left[F(\xi + y\sigma_n) - \tfrac{1}{2}\right]^2\right\}}. \end{aligned}$$

It can be shown (by differentiations) that $h_1(t)$ and $h_2(t)$ are respectively monotonically increasing and decreasing functions of t, when $t \geqq 0$, and that $\lim_{t \to 0} h_1(t) = \lim_{t \to 0} h_2(t) = 1$. Hence we obtain from (1) and (2)

$$H(y) - \tfrac{1}{2} \geqq aB_n \sqrt{1 - \frac{1}{2n+2}} \left[\phi\left(t_1 \sqrt{\frac{2n+2}{2n+1}}\right) - \tfrac{1}{2}\right],$$

$$H(y) - \tfrac{1}{2} \leqq bB_n \sqrt{1 + \frac{1}{2n}} \left[\phi\left(t_2 \sqrt{\frac{2n}{2n+1}}\right) - \tfrac{1}{2}\right],$$

where

$$(5) \qquad \phi(t) = \int_{-\infty}^{t} (1/\sqrt{2\pi}) \exp(-\tfrac{1}{2}x^2)\, dx.$$

In a similar way we can show that, for arbitrary $x > 0$ and $y > 0$,

$$(6)\quad \begin{aligned} &H(y) - H(-x) \\ &\quad \geqq aB_n \sqrt{1 - \frac{1}{2n+2}} \left[\phi\left(t_1 \sqrt{\frac{2n+2}{2n+1}}\right) - \phi\left(-t_3 \sqrt{\frac{2n+2}{2n+1}}\right)\right], \\ &H(y) - H(-x) \\ &\quad \leqq bB_n \sqrt{1 + \frac{1}{2n}} \left[\phi\left(t_2 \sqrt{\frac{2n}{2n+1}}\right) - \phi\left(-t_4 \sqrt{\frac{2n}{2n+1}}\right)\right], \end{aligned}$$

where t_3 and t_4 are obtained from t_1 and t_2 in (4) by replacing y by $-x$.

It can be seen from (4) that if x and y are fixed, $t_1 = t_2 = y + O(1)$ and $t_3 = t_4 = x + O(1)$ for large n. The following sections will show, for various kinds of parent distributions, that if a and b are properly chosen, then (6) remains valid if t_1, t_2 and t_3, t_4 are replaced by y and x, respectively.

If n is large, upper and lower bounds for B_n can be obtained by using Stirling's formula. Feller [3] showed that for $n \geqq 4$,

$$n! = \sqrt{2\pi}\, n^{n+1/2} \exp\left[-n + \frac{1}{12n} - \frac{(1+\theta)}{360n^3}\right], \qquad |\theta| \leqq \tfrac{1}{6}.$$

If the last term, $-(1 + \theta) / 360n^3$, is omitted, then it can be shown that

$$1 + \frac{1}{8n} - \frac{7n+3}{24n^2(2n+1)} < B_n < 1 + \frac{1}{8n} + \frac{1}{16n(8n-1)},$$

or $B_n \sim 1 + 1/8n$.

4. Normal parent population. Suppose that a sample of size $2n + 1$ is drawn from a normal population with mean ξ and variance σ^2. The distribution of $\tilde{x}$ is then asymptotically normal with mean ξ and variance $\pi\sigma^2 / 2(2n + 1)$. It has been shown that if, for $x > 0$,

$$(7)\qquad \phi(x) - \phi(-x) = a(x) \sqrt{1 - \exp\left[-(2/\pi)\, x^2\right]},$$

then $a(x)$, a function of x, never exceeds 1 and is very close to 1 for all values of $x > 0$. Williams [6] proved that $a(x) \leqq 1$ and tabulated $1/a(x) - 1$ for a number of values of x ranging from .1 to 2.0. Pólya [5] gave several proofs for the same inequality and remarked that if $\sqrt{1 - \exp\left[-(2/\pi)x^2\right]}$ is used as an approximation to $\phi(x) - \phi(-x)$, "then the error committed is less than one per cent (even less than .71 per cent) of the quantity approximated." In other words, $a(x) > .9929$ for all $x > 0$.

For arbitrary $x > 0$ and $y > 0$, let

$$x_n = \sqrt{\pi/2}\;\; x / \sqrt{2n+1}, \qquad y_n = \sqrt{\pi/2}\;\; y / \sqrt{2n+1}.$$

Applying (7) to (6) yields

$$H(y) - H(-x) \geqq \min\{a(x_n), a(y_n)\} \cdot B_n \sqrt{1 - \frac{1}{2n+2}} \left[\phi\left(y\sqrt{\frac{2n+2}{2n+1}}\right) - \phi\left(-x\sqrt{\frac{2n+2}{2n+1}}\right)\right], \tag{8}$$

$$H(y) - H(-x) \leqq B_n \sqrt{1 + \frac{1}{2n}} \left[\phi\left(y\sqrt{\frac{2n}{2n+1}}\right) - \phi\left(-x\sqrt{\frac{2n}{2n+1}}\right)\right], \tag{9}$$

where $\phi(x)$ and $a(x)$ are defined by (5) and (7). If no very high accuracy is required one may use

$$H(y) - H(-x) \geqq .9929\,(1 + \tfrac{1}{8}n)\sqrt{1 - 1/(2n+2)}\,[\phi(y) - \phi(-x)], \tag{10}$$

$$H(y) - H(-x) \leqq (1 + \tfrac{1}{8}n)\sqrt{1 + 1/2n}\,[\phi(y) - \phi(-x)]. \tag{11}$$

5. Other parent populations.

A. Rectangular distribution. Let $f(x) = 1/(d-c)$, where $c < x < d$. Then $\xi = \frac{1}{2}(c+d)$ and $\sigma_n^2 = (c-d)^2/4(2n+1)$. Let $H(x)$ be the cdf of $(\bar{x} - \xi)/\sigma_n$. If $x > 0$ and $y > 0$, then lower bounds for $H(y) - H(-x)$ are the right sides of (8) and (10), without the factors min $\{a(x_n), a(y_n)\}$ and .9929. The upper bounds for $H(y) - H(-x)$ are the right sides of (9) and (11) with an additional factor max $\{b(x_n), b(y_n)\}$, where $b(x)$ is defined by

$$b(x) = \sqrt{\frac{x}{1 - e^{-x}}}, \quad x > 0; \qquad x_n = \frac{x^2}{2n+1}; \qquad y_n = \frac{y^2}{2n+1}.$$

We note that $b(x)$ is close to 1 only if x is close to 0, for example, $b(.1) = 1.02$ and $b(.2) = 1.05$. This means that unless x and y are small, upper and lower bounds for $H(y) - H(-x)$ are not very close to each other except for large n.

B. Laplace distribution. Let $f(x) = (1/2\lambda)\exp[-|x-\xi|/\lambda]$, where $-\infty < x < \infty$. Then ξ is the median and $\sigma_n^2 = \lambda^2/(2n+1)$. Define $c(x)$ by

$$1 - e^{-x} = c(x)\sqrt{1 - e^{-x^2}}, \qquad x > 0. \tag{12}$$

We say that $c(x) \leqq 1$. If $x \geqq 1$, this is obvious; if $x \leqq 1$, we use (7) to prove it. It then becomes clear that upper bounds for $H(y) - H(-x)$ are the same as those corresponding to a normal parent population, that is (9) and (11). Lower bounds for $H(y) - H(-x)$ are the right sides of (8) and (10) with min $\{a(x_n), a(y_n)\}$ and .9929 replaced by min $\{c(x_n), c(y_n)\}$, where $c(x)$ is defined by (12) and $x_n = x/\sqrt{2n+1}$. Again we remark that $c(x)$ is close to 1 if x is small or large, for example, $c(.1) = .99$, $c(.2) = .92$, and $c(3) = .97$, while $c(1) = .79$ and $c(2) = .87$.

Finally we note that since $c(x)$ tends to 1 as x tends to 0, $H(y) - H(-x)$ tends to $\phi(y) - \phi(-x)$ as n tends to infinity. Therefore $(\bar{x} - \xi)/\sigma_n$ has an asymptotically normal distribution. In the general theorem which showed that $(\bar{x} - \xi)/\sigma_n$

has an asymptotically normal distribution (see] 2], p. 369, also the beginning of Sec. 3), it is required that $f'(\xi)$ be continuous. For a Laplace distribution, however, $f'(\xi)$ does not exist.

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SOME NEW TEST CRITERIA IN MULTIVARIATE ANALYSIS

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1. Summary. Three new test criteria are proposed for overall tests of hypotheses in multivariate analysis. They are based on the characteristic roots of certain matrices obtained from the product moment matrices of samples drawn from multivariate normal populations. The approximate distributions of the statistics involved in the tests are found as Type I or Type II Beta distributions.

2. Introduction. In multivariate analysis, we generally wish to test three hypotheses, namely

(I) that of equality of the dispersion matrices of two p-variate normal populations,

(II) that of equality of the p-dimensional mean vectors for l p-variate normal populations (which is mathematically identical with the general problem of multivariate analysis of variance of means); and

(III) that of independence between a p-set and a q-set of variates in a $(p + q)$-variate normal population, with $p \leqq q$.

All tests proposed so far for these hypotheses have been shown to depend, when the hypotheses to be tested are true, only on the characteristic roots of matrices based on sample observations. For example, in case (I), all the tests proposed so far are based on the characteristic roots of the matrix $S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 denote the usual sum of product (S.P.) matrices and where both are almost everywhere positive definite (a.e. p.d.). Thus $S_1(S_1 + S_2)^{-1}$ is a.e., p.d., whence it follows that all the p characteristic roots are greater than zero and less than unity. In case (II), the matrix is $S^*(S^* + S)^{-1}$, where S^* denotes the "between" S.P. matrix of means weighted by the sample sizes and S denotes the "within" S.P. matrix (pooled from the S.P. matrices of l samples). Then S is a.e. p.d., and S^* is at least positive semidefinite of rank $s = \min(p, \ l - 1)$. Thus, a.e., s of the characteristic roots are greater than zero and less than unity and the $p - s$ remaining roots are zero. In case (III), the matrix is $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where S_{11} is the S.P. matrix of the sample of observations on the p-set of variates, S_{22} that on the q-set, and S_{12}, the S.P. matrix between the observations on the p-set and those on the q-set. If $p \leqq q$ and $p + q < k$, where k is the sample size, then a.e. the p characteristic roots of this matrix are greater than zero and less than unity.

In each case, if the hypothesis to be tested is true, the $s \leqq p$ nonzero roots θ_i, where $0 < \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_s < 1$, have the same joint distribution, the form of which was given by Roy [7], Hsu [3], and Fisher [1]. The distribution

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can be written in the form

$$(1)\qquad p(\theta_1, \cdots, \theta_s) = C(s, m, n,) \prod_{i=1}^{s} \theta_i^m (1 - \theta_i)^n \prod_{i>j} (\theta_i - \theta_j),$$

$$0 < \theta_1 \leqq \cdots \leqq \theta_s < 1,$$

where

$$C(s, m, n) = \frac{\pi^{s/2} \prod_{i=1}^{s} \Gamma\tfrac{1}{2}(2m + 2n + s + i + 2)}{\prod_{i=1}^{s} \Gamma\tfrac{1}{2}(2m + i + 1)\, \Gamma\tfrac{1}{2}(2n + i + 1)\, \Gamma\tfrac{1}{2}i}.$$

Here m and n are to be interpreted differently for the different situations. For example, in case (I), with n_1 and n_2 as the sample sizes,

$$(2)\qquad m = \tfrac{1}{2}(n_1 - p - 2), \qquad n = \tfrac{1}{2}(n_2 - p - 2).$$

In case (II), with N the total of the sizes of l samples,

$$(3)\qquad m = \tfrac{1}{2}(|l - p - 1| - 1), \qquad n = \tfrac{1}{2}(N - l - p - 1).$$

In case (III),

$$(4)\qquad m = \tfrac{1}{2}(q - p - 1), \qquad n = \tfrac{1}{2}(k - p - q - 2).$$

3. Current test criteria. The test criteria previously proposed for tests of hypotheses in multivariate analysis are mainly

Hotelling's $T_0^2 = c \sum_1^s \lambda_i$, where $\lambda_i = \theta_i / (1 - \theta_i)$ and c is a constant depending on the degrees of freedom of a matrix from which the characteristic roots were obtained [2];

Roy's [8], [9], [10] criteria of the largest root $\theta_s(\lambda_s)$ and the smallest root $\theta_1(\lambda_1)$;

Wilks' [11] criterion, $\Lambda = \prod_1^s (1 - \theta_i)$; and

$V^{(s)} = \sum_1^s \theta_i$, [4], [5].

It is well known (see e.g., [9], [10]) that (i) the sample roots are invariant under certain classes of linear transformations, different for the different situations discussed in the Introduction, and (ii) the joint distribution of these roots, when the hypothesis to be tested is true, involves as parameters only the so-called degrees of freedom, and, in addition, only a set of population roots (with a strong physical import) when the hypothesis to be tested is not true.

It is likely that any statistic (and a test based on it), in order to possess these desirable properties, must be a function of these sample roots. All the tests proposed so far, including the ones proposed in this paper, are, in fact, based on functions of these roots and therefore have all the above properties.

The choice of any specific function of the roots, as a basis for test criteria, has so far been made on additional considerations which are heuristic. The ultimate

justification is found in properties like (iii) usability in the sense of availability of large or small sample distribution of the statistic involved (on the null hypothesis) and (iv) some good operating characteristics of the test. Such characteristics are, for example, (iva) unbiassedness and (ivb) even something stronger, namely a power which is a monotonically increasing function of each of the deviation parameters, (ivc) some good and convenient lower bound to the power, (ivd) possibility of getting, by inversion, suitable confidence interval statements, and (ive) admissibility.

Hotelling's T_0^2 has so far been studied and shown to be good under (i)–(iii). Wilks' criterion has, as of moment, been shown to be good under (i)–(iii) and also under (iva)–(ivb), in so far as the criterion is to be used for situations (II) and (III) discussed in the Introduction. Roy's criteria have so far been shown to be good under (i)–(iii) and (iva)–(ivd). The tests proposed in the present paper are also based on heuristic arguments, and are shown to be good under (i)–(iii). The further study of these tests in terms of (iva)–(ive) remains to be made.

4. Three new test criteria. Wilks' criterion is the sth power of a geometric mean, while T_0^2 and $V^{(s)}$ are s times the arithmetic means. This suggests the harmonic mean as another possibility. The following three criteria based on this statistic are offered.

$$H^{(s)} = s\left\{\sum_{i=1}^{s}(1-\theta_i)^{-1}\right\}^{-1}, \quad R^{(s)} = s\left\{\sum_{i=1}^{s}\theta_i^{-1}\right\}^{-1}, \quad T^{(s)} = s\left\{\sum_{i=1}^{s}\lambda_i^{-1}\right\}^{-1}.$$

Before proceeding to derive the approximate distributions of these criteria, we may derive the approximate distribution of the criterion defined by $W^{(s)} = 1 - V^{(s)}/s$, the tests based on which are exactly identical with those of $V^{(s)}$. It has been shown [4], [5] that the distribution of $V^{(s)}$ can be approximated by a Type I Beta distribution given by

$$(5) \qquad p(V^{(s)}) = C\{V^{(s)}\}^{s(2m+s+1)/2-1}(1 - V^{(s)}/s)^{s(2n+s+1)/2-1}, \qquad 0 < V^{(s)} < s,$$

where $C = 1 \,/\, s^{s(2m+s+1)/2}\beta\{\frac{1}{2}s(2m + s + 1), \ \frac{1}{2}s(2n + s + 1)\}$. Hence it is easily seen that $W^{(s)}$ has the approximate distribution

$$(6) \qquad p(W^{(s)}) = C'\{W^{(s)}\}^{s(2n+s+1)/2-1}(1 - W^{(s)})^{s(2m+s+1)/2-1}, \qquad 0 < W^{(s)} < 1,$$

where $C' = 1/\beta\{\frac{1}{2}s(2n + s + 1), \ \frac{1}{2}s(2m + s + 1)\}$. The approximate distribution (5) has been shown, [5], to be satisfactory for practical use for $m + n \geqq 30$ when $s = 2$; when s increases by one unit, $m + n$ must increase by 10 to give satisfactory results. Hence under these conditions (6) is also valid.

Now we consider the criterion $H^{(s)}$. If we put $U^{(s)} = T_0^2/c$, the approximate distribution of $U^{(s)}$ has been shown [5], [6] to be a Type II Beta distribution of the form

$$(7) \qquad p(U^{(s)}) = K\{U^{(s)}\}^{s(2m+s+1)/2-1} \,/\, [1 + U^{(s)}/s]^{s(2m+2n+s+1)/2+1}, \qquad 0 < U^{(s)} < \infty,$$

where $K = 1/s^{s(2m+s+1)/2}\beta\{\frac{1}{2}s(2m+s+1),\ sn+1\}$. It may be seen that

$$U^{(s)} = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \theta_i/(1-\theta_i) = \sum_{i=1}^{s} (1-\theta_i)^{-1} - s. \tag{8}$$

Hence $H^{(s)} = (1 + U^{(s)}/s)^{-1}$, and use of this transformation in (7) gives

$$p(H^{(s)}) = \{H^{(s)}\}^{sn}(1-H^{(s)})^{s(2m+s+1)/2-1} / \beta\{sn+1,\ \tfrac{1}{2}s(2m+s+1)\}, \tag{9}$$
$$0 < H^{(s)} < 1.$$

For this approximation to be valid it has been shown [5] that $n + s$ must satisfy the conditions stated for $m + n$ in the case of $V^{(s)}$.

Again, to obtain the distribution of $R^{(s)}$, if we make a transformation $\theta_i' = 1 - \theta_i$, for $i = 1, 2, \cdots, s$, the joint distribution of θ_i''s resulting from (1) will have the same form as the joint distribution of θ_i's with m and n interchanged. Also we have $0 < \theta_s' \leqq \cdots \leqq \theta_1' < 1$. Hence if we start with the distribution of θ_i''s, we obtain for

$$U'^{(s)} = \sum_{i=1}^{s} \theta_i'/(1-\theta_i')$$

the same form of approximate distribution as the one given by (7) with m and n interchanged. Now we consider

$$U'^{(s)} = \sum_{i=1}^{s} \theta_i^{-1} - s. \tag{10}$$

Hence $R^{(s)} = (1 + U'^{(s)}/s)^{-1}$, and with this transformation we have, as in (9) the approximate distribution of $R^{(s)}$ in the form

$$p(R^{(s)}) = \{R^{(s)}\}^{sm}(1-R^{(s)})^{s(2n+s+1)/2-1} / \beta\{sm+1,\ \tfrac{1}{2}s(2n+s+1)\}, \tag{11}$$
$$0 < R^{(s)} < 1.$$

Here $m + s$ must satisfy the conditions stated for $m + n$ in the case of $V^{(s)}$, for the validity of the approximation.

It is interesting to note the parallelisms between the approximate distributions of $V^{(s)}$ and $W^{(s)}$, and between those of $H^{(s)}$ and $R^{(s)}$. One is obtainable from the other by an interchange of m and n. It may be further noted that

$$U'^{(s)} = \sum_{i=1}^{s} \lambda_i^{-1} = \sum_{i=1}^{s} \theta_i^{-1} - s \tag{12}$$

Hence $T^{(s)} = R^{(s)} / (1 - R^{(s)})$, and from (11), the distribution of $T^{(s)}$ can be obtained in the form

$$p(T^{(s)}) = \frac{\{T^{(s)}\}^{sm}}{(1+T^{(s)})^{s(2m+2n+s+1)/2+1}\beta\{sm+1,\ \frac{1}{2}s(2n+s+1)\}}, 0 < T^{(s)} < \infty. \tag{13}$$

The approximate distribution $p(T^{(s)})$ also must satisfy the conditions stated for $p(R^{(s)})$ in (11). The condition that m must be large is more often suited to case (I) of the Introduction than to cases (II) and (III).

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DISTRIBUTION OF DEFINITE AND OF INDEFINITE QUADRATIC FORMS

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1. Summary. A previous paper [1] has given a method of approximating the distribution of a quadratic form in normally distributed variables by means of convergent Laguerrian expansions. In the case of an indefinite quadratic form, however, the method was restrictive in that it might be difficult to obtain the semi-moments required in computing the coefficients of the expansion. The present article circumvents this difficulty for positive values of the argument of the distribution function, when the number of positive or the number of negative eigenvalues is even, and also yields convergent expansions for the distribution function involving Laguerre polynomials. The proposed method has the further advantage that no moments or semi-moments need be calculated.

2. Introduction. Suppose the random variable $X = (X_1, X_2, \cdots, X_n)$ has the probability density function

$$p(x) = (2\pi)^{-n/2} |\Omega|^{1/2} \exp(-\tfrac{1}{2}x\Omega x'), \tag{1}$$

where $x\Omega x'$ is a positive definite quadratic form. It is required to find the distribution of XQX', where Q is any $n \times n$ symmetric matrix. A linear transformation permits the problem to be posed as follows. Find the distribution function $F(x)$ of $\sum_1^n \lambda_i X_i^2$, where the λ's are real numbers and X has the probability density function

$$f(x) = (2\pi)^{-n/2} \exp(-\tfrac{1}{2}xx'). \tag{2}$$

A special case of this problem, with the λ's positive and satisfying the restriction (in the above notation)

$$\lambda_k < \frac{2}{n}\sum_1^n \lambda_j, \qquad k = 1, 2, \cdots, n, \tag{3}$$

was considered[1] by Bhattacharya [2], who expressed the probability density as a convergent series in Laguerre polynomials. The present article employs a similar but more general method for expanding the cumulative distribution function, without restriction (3) and also without the restriction that all the λ's be positive.

3. Distribution of a positive-definite quadratic form. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all positive, and set

$$\alpha_i = \lambda_i - \bar{\lambda}, \tag{4}$$

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[1] The author is grateful to L. Herbach for bringing this reference to his attention.

where $\tilde{\lambda}$ is an arbitrary number satisfying the inequality

$$\tilde{\lambda} > \tfrac{1}{2} \max_i \lambda_i. \tag{5}$$

The characteristic function of $\sum_1^n \lambda_i X_i^2$ may be written as

$$\phi(t) = (1 - 2i\tilde{\lambda}t)^{-n/2} \prod_{j=1}^{n} \left(1 - \frac{2i\alpha_j t}{1 - 2i\tilde{\lambda}t}\right)^{-1/2}.$$

Since

$$\left|\frac{2i\alpha_j t}{1 - 2i\tilde{\lambda}t}\right| < 1, \qquad j = 1, 2, \cdots, n$$

for all values of t, $\phi(t)$ may be expanded as the product of n power series. Thus

$$\phi(t) = \sum_{k=0}^{\infty} a_k(-2it)^k(1 - 2i\tilde{\lambda}t)^{-n/2-k}, \tag{6}$$

where a_k is the coefficient of r^k in the expansion

$$\prod_{j=1}^{n} \sum_{i=1}^{\infty} \alpha_j^i \beta_i r^i, \qquad \beta_i = \frac{(-\tfrac{1}{2})(-\tfrac{1}{2} - 1) \cdots (-\tfrac{1}{2} - i + 1)}{i!} = (-\tfrac{1}{4})^i \binom{2i}{i}$$

Explicitly, a_k may be written as

$$\begin{aligned} a_k = {} & \beta_k \sum_{i=1}^{n} \alpha_i^k + \beta_{k-1}\beta_1 \sum_{i<j} \sum \alpha_i^{k-1}\alpha_j + \beta_{k-2}\beta_2 \sum_{i<j} \sum \alpha_i^{k-2}\alpha_j^2 \\ & + \beta_{k-2}\beta_1^2 \sum_{i<j<l} \sum \sum \alpha_i^{k-2}\alpha_j\alpha_l + \beta_{k-3}\beta_2\beta_1 \sum_{i<j<l} \sum \sum \alpha_i^{k-3}\alpha_j^2\alpha_l + \cdots. \end{aligned} \tag{7}$$

Thus

$$a_0 = 1, \qquad a_1 = \beta_1 \sum_1^n \alpha_i, \qquad a_2 = \beta_2 \sum_1^n \alpha_i^2 + \beta_1^2 \sum_{i \neq j} \sum \alpha_i\alpha_j,$$

$$a_3 = \beta_3 \sum_1^n \alpha_i^3 + \beta_2\beta_1 \sum_{i \neq j} \sum \alpha_i^2\alpha_j + \beta_i^3 \sum_{i \neq j \neq l} \sum \sum \alpha_i\alpha_j\alpha_l,$$

and so on. Application of the inversion formula [3]

$$F(x) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\phi(t)e^{-itx}}{t}\, dt, \tag{8}$$

noting that the series in (6) is uniformly convergent for all values of t, yields

$$F(x) = \frac{1}{2} - \frac{1}{2\pi i} \sum_{k=0}^{\infty} a_k \oint \frac{(-2it)^k(1 - 2i\tilde{\lambda}t)^{-n/2-k}}{t}\, e^{-itx}\, dt.$$

These integrals may be evaluated by use of the following identity in x, obtained by applying the inversion formula (8) to the characteristic function of a χ^2 distribution with $2k + n$ degrees of freedom.

$$\frac{1}{2} - \frac{1}{2\pi i} \oint \frac{e^{-2\tilde{\lambda}tx}(1 - 2i\tilde{\lambda}t)^{-n/2-k}}{t}\, dt = \frac{1}{2^{n/2+k}\Gamma(n/2 + k)} \int_0^{2x} v^{n/2+k-1}e^{-v/2}\, dv.$$

This is differentiated k times, yielding

$$\frac{(\bar{\lambda})^k}{-2\pi i}\oint \frac{(-2it)^k(1-2i\bar{\lambda}t)^{-n/2-k}e^{-2i\bar{\lambda}tx}}{t}\,dt = \Gamma(n/2+k)\left(\frac{d}{dx}\right)^{k-1} e^{-x}x^{n/2+k-1},$$

which is valid for $k \geqq 1$ and for all values of $x \geqq 0$, since the integral on the left side converges uniformly. But from the theory of Laguerre polynomials [4]

$$\left(\frac{d}{dx}\right)^m e^{-x}x^\gamma = m!e^{-x}x^\gamma L_m^{(\gamma)}(x), \qquad \gamma > -1 \tag{9}$$

Hence

$$-\frac{1}{2\pi i}\oint \frac{(-2it)^k(1-2i\bar{\lambda}t)^{-n/2-k}e^{-2i\bar{\lambda}tx}}{t}\,dt = \frac{\Gamma(k)}{\Gamma(k+n/2)}\,\frac{e^{-x}x^{n/2}L_{k-1}^{(n/2)}}{\bar{\lambda}^k}$$

and the distribution function of $\sum_1^n \lambda_i X_i^2$ may be written

$$(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^{x/\bar{\lambda}} v^{n/2-1}e^{-v/2}\,dv + \sum_{k=1}^{\infty} a_k \frac{\Gamma(k)}{\Gamma(k+n/2)}\,\frac{e^{-x/2\bar{\lambda}}x^{n/2}L_{k-1}^{(n/2)}(x/2\bar{\lambda})}{2^{n/2}\bar{\lambda}^{k+n/2}}$$

$$x \geqq 0.$$

It is, of course, evident that a similar argument can be used to obtain an analogous expression for the probability density, $F'(x)$.

4. The difference of two independent χ^2 random variables. Before considering the distribution of an indefinite quadratic form in the next section, it is necessary to refer to the distribution of a difference of two independent χ^2 random variables. Let X and Y have the probability density function

$$p(x, y) = (1/c)e^{-(x+y)/2}x^{(f_1/2)-1}y^{(f_2/2)-1}, \qquad x > 0, \quad y > 0,$$

where $c = 2^{(f_1+f_2)/2}\Gamma(f_1/2)\Gamma(f_2/2)$. The joint probability density function of Y and $V = X - Y$ is

$$(1/c)e^{-v/2-y}(y+v)^{f_1/2-1}y^{f_2/2-1}, \qquad y > 0, \quad v + y > 0.$$

If it is assumed that f_1 is an even integer, say $2m$, then we may apply the binomial expansion and integrate out y to obtain the probability density of V.

$$p(v) = \begin{cases} \dfrac{1}{c}\displaystyle\sum_{h=0}^{m-1}\binom{m-1}{h}\Gamma\left(h+\frac{f_2}{2}\right)e^{-v/2}v^{m-1-h}, & v \geqq 0 \\ \dfrac{1}{c}\displaystyle\sum_{h=0}^{m-1}\binom{m-1}{h}e^{-v/2}v^{m-1-h}\int_{-v}^{\infty} e^{-y}y^{h+f_2/2-1}\,dy & v \leqq 0. \end{cases}$$

Let $K = \displaystyle\int_{-\infty}^{0} p(v)\,dv = [1/B(f_1/2, f_2/2)]\int_{1/2}^{1}(1-w)^{f_1/2-1}w^{f_2/2-1}\,dw$, which is available from tables of the incomplete Beta function. Now we apply inversion

formula (8) to the density $p(v)$ to obtain

$$\frac{1}{2} - \frac{1}{2\pi i}\oint \frac{e^{-2i\tilde{\lambda}tz}}{(1-2i\tilde{\lambda}t)^{f_1/2}(1+2i\tilde{\lambda}t)^{f_2/2}}\frac{dt}{t}$$

$$= \begin{cases} K + \dfrac{1}{c}\displaystyle\sum_{h=0}^{m-1}\binom{m-1}{h}\Gamma(h+f_2/2)\int_0^{2x} e^{-v/2}v^{m-1-h}\,dv, & x \geqq 0; \quad (10) \\ \dfrac{1}{c}\displaystyle\sum_{h=0}^{m-1}\binom{m-1}{h}\int_{-\infty}^{2x} e^{-v/2}v^{m-1-h}\,dv\int_v^{\infty} e^{-y}y^{h+f_2/2-1}\,dy, & x \leqq 0. \quad (11) \end{cases}$$

First, in the case $x \geqq 0$, the kth derivative with respect to x of (10) yields

$$\frac{-1}{2\pi i}\oint \frac{e^{-2i\tilde{\lambda}tz}(-2i\tilde{\lambda}t)^k}{(1-2i\tilde{\lambda}t)^{f_1/2}(1+2i\tilde{\lambda}t)^{f_2/2}}\frac{dt}{t}$$

$$= \frac{1}{c}\sum_{h=0}^{m-1}\binom{m-1}{h}2^{m-h}\Gamma(h+f_2/2)e^{-x}K^{(x)}_{m-1-h,\ k-1}, \qquad k \geqq 1, \quad x \geqq 0, \quad (12)$$

where $K_{p,q}(x)$ is a polynomial of degree p defined by $(d/dx)^q e^{-x}x^p = e^{-x}K_{p,q}(x)$ From the definition of Laguerre polynomials (cf. (9) above), the K-polynomials have the relation[2] to Laguerre polynomials

$$K_{p,q}(x) = q!x^{p-q}L_q^{(p-q)}(x), \qquad p - q > -1$$

The restriction $p - q > -1$ is no drawback, however, since $x^qK_{p,q}(x) = x^pK_{q,p}(x)$.

Now, in the case $x \leqq 0$, we assume that f_2 is an even integer, $2m'$ say. Before differentiating (11) we note that

$$\frac{d}{dx}\left\{\int_{-\infty}^{2x} e^{-v/2}v^{m-1-h}\,dv\int_v^{\infty} e^{-y}y^{h+m'-1}\,dy\right\} = e^{-x}(2x)^{m-1-h}\int_{-2x}^{\infty} y^{h+m'-1}\,dy,$$

and define $J_{p,q}(x)$ by $(d/dx)^q\int_{-2x}^{\infty} e^{-y}y^p\,dy = e^{2x}J_{p,q}(x)$. (For $q \geqq 1$, $J_{p,q}$ is a polynomial of degree p.) Then the kth derivative of (11) can be written as

$$-\frac{1}{2\pi i}\oint \frac{e^{-2i\tilde{\lambda}tz}(2i\tilde{\lambda}t)^k}{(1-2i\tilde{\lambda}t)^{f_1/2}(1+2i\tilde{\lambda}t)^{f_2/2}}\cdot\frac{dt}{t}$$

$$= \frac{e^x}{c}\sum_{h=0}^{m-1}\binom{m-1}{h}\sum_{r=0}^{k-1}\left[\binom{k-1}{r}K^{(x)}_{m-1-h,\ k-1-r}J^{(x)}_{h+m'-1,\ r}\right]. \quad (13)$$

The J-polynomials are related to the K-polynomials (and hence to the Laguerre polynomials) by

$$J_{p,q}(x) = (-2)^p e^{2x}\sum_{s=0}^{q-1} 3^{q-1-s}\binom{q-1}{s}K_{p,s}(x).$$

[2] This relation was kindly pointed out to the author by the referee.

5. Distribution of an indefinite quadratic form. Suppose

$$XQX' = \sum_{1}^{n_1} \lambda_i X_i^2 - \sum_{n_1+1}^{n_1+n_2} \lambda_i X_i^2,$$

where $\lambda_i > 0$ for $i = 1, 2, \cdots, n$, and $n = n_1 + n_2$. We may assume that X has the probability density $f(x)$ of (2). Define α_j and $\bar{\lambda}$ as in (4) and (5), respectively. Then the characteristic function $\phi(t)$ of XQX' can be written as

$$(1 - 2it\bar{\lambda})^{-n_1/2}(1 + 2it\bar{\lambda})^{-n_2/2} \prod_{j=1}^{n_1}\left(1 - \frac{2it\alpha_j}{1 - 2it\bar{\lambda}}\right)^{-1/2} \prod_{j=n_1+1}^{n_1+n_2}\left(1 + \frac{2it\alpha_j}{1 + 2it\bar{\lambda}}\right)^{-1/2}.$$

As in Section 3, this may be expanded as a product of n power series to give

$$\phi(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j}(-2it)^j(1 - 2it\bar{\lambda})^{-n_1/2-j}(2it)^{k-j}(1 + 2it\bar{\lambda})^{-n_2/2-k+j}$$

which is uniformly convergent for all values of t. Here a_k is expressible as in (7) with n_1 replacing n. Analogously, b_k may be expressed as

$$b_k = \beta_k \sum_{i=n_1+1}^{n_1+n_2} \alpha_i^k + \beta_{k-1}\beta_1 \sum_{i<j}\sum \alpha_i^{k-1}\alpha_j + \beta_{k-2} \sum_{i<j}\sum \alpha_i^{k-2}\alpha_j^2 + \cdots$$

for $i = n_1 + 1, \quad n_1 + 2, \cdots, \quad n_1 + n_2$ and $j = n_1 + 1, \quad n_1 + 2, \cdots, \quad n_1 + n_2$. Applying the inversion formula (8), the distribution function may be written

$$F(x) = \frac{1}{2} - \frac{1}{2\pi i} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j a_j b_{k-j} \oint \frac{(2it)^k e^{-itx}}{(1 - 2it\bar{\lambda})^{n_1/2+j}(1 + 2it\bar{\lambda})^{n_2/2+k-j}} \frac{dt}{t}.$$

In virtue of (12) this becomes, for $x \geqq 0$,

$$F(x) = K + \frac{1}{c} \sum_{h=0}^{m-1} \binom{m-1}{h} \Gamma\left(h + \frac{n_2}{2}\right) \int_0^{x/\bar{\lambda}} e^{-v/2} v^{m-1-h}\, dv$$

$$+ \frac{e^{-x/2\bar{\lambda}}}{c} \sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^{j+k} a_j b_{k-j} \bar{\lambda}^{-k} \sum_{h=0}^{m-1} \binom{m-1}{h}$$

$$\cdot 2^{m-h} \Gamma\left(h + \frac{n_2}{2}\right) K_{m-1-h,\ k-1}\left(\frac{x}{2\bar{\lambda}}\right),$$

where n_1 is an even integer, say $2m$.

For $x \leqq 0$, (13) is applied, with $n_2 = 2m'$, to give

$$F(x) = \frac{1}{c} \sum_{h=0}^{m-1} \binom{m-1}{h} \int_{-\infty}^{x/\lambda} e^{-v/2} v^{m-1-h}\, dv \int_v^{\infty} e^{-y} y^{h+m'-1}\, dy$$

$$+ \frac{e^{-x/2\bar{\lambda}}}{c} \sum_{k-1}^{\infty} \sum_{j=0}^{k} (-1)^{j+k} a_j b_{k-j} \bar{\lambda}^{-k} \sum_{h=0}^{m-1} \sum_{r=0}^{k-1} \binom{m-1}{h} \binom{k-1}{r}$$

$$\cdot K_{m-1-h,\ k-1-r}\left(\frac{x}{2\bar{\lambda}}\right) J_{h+m'-1,\ r}\left(\frac{x}{2\bar{\lambda}}\right).$$

The additional assumption $n_2 = 2m'$, stated above, is needed here only for the purpose of evaluating

$$\int_{-\infty}^{-x/\tilde{\lambda}} e^{-x/2} v^{m-1-h}\, dv \int_{v}^{\infty} e^{-y} y^{h+m'-1}\, dy = I, \quad \text{say.}$$

By change of variables we may write

$$I = \int_{-\infty}^{-x/\tilde{\lambda}} e^{-v/2} v^{m-1-h}\, dv \int_{0}^{\infty} e^{-(z-v)} (z - v)^{h+m'+1}\, dz$$

$$= \sum_{t=0}^{h+m'-1} (-1)^t \binom{h + m' - 1}{t} \Gamma(h + m' - t) \int_{-\infty}^{-x/\tilde{\lambda}} e^{v/2} v^{t+m-1}\, dv.$$

This can now be evaluated with the aid of tables of the incomplete Gamma function.

6. Conclusion. Although the distribution function of a positive definite quadratic form is approximated by a relatively simple Laguerrian expansion, the distribution of an indefinite quadratic form gives rise to a more complicated expansion involving Laguerre polynomials. This is, of course, due mainly to the fact that the weight function corresponding to the orthogonal system of Laguerre polynomials is zero for negative values of the argument. On the other hand, Gram-Charlier series, although convenient for asymptotic expansion theory, do not converge for a sufficiently wide class of distribution functions (cf [1]).

The present article has dealt essentially with the question of convergence, and has shown how to construct a series involving Laguerre polynomials which actually converges to the distribution function. An important question not considered in this paper is how rapidly these Laguerrian expansions converge. Some results along these line are now being prepared for publication.

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NOTE ON THE DISTRIBUTION OF A DEFINITE QUADRATIC FORM[1, 2, 3]

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1. Summary. An expression is derived for the distribution of a definite quadratic form in independent $N(0, 1)$ variates which depends only on the value of the determinant of the form and on the moments of a quadratic form whose matrix is the inverse of the original quadratic form. This expression is an alternating series which converges absolutely and is such that if we stop with any even power of the series we have an upper bound and if we stop with any odd power of the series a lower bound to the cumulative distribution function. The result given in this note seems to be in several ways an improvement over the method given in Robbins [2].

2. Notation. All vectors are column vectors and primes indicate their transposes. Thus $X' = [X_1, \cdots, X_n]$. In addition dx_n stands for $dx_1 \cdots dx_n$, while $|A|$ stands for the product $a_1 \cdots a_n$ where $a_1, \cdots, a_n$ are the latent roots of the matrix A, and $E(Q_n)^k$ stands for the kth moment of Q_n.

3. The problem. Suppose we have a quadratic form $Q_n = \frac{1}{2}Y'AY$ in $Y_1, \cdots, Y_n$ where the Y_i are independent $N(0, 1)$ variates, and where A denotes a definite $n \times n$ matrix. It is well known that we can make an orthogonal transformation reducing Q_n to its canonical form, that is $Q_n = \frac{1}{2}\sum a_i X_i^2$ where the elements $a_1, \cdots, a_n$ are the latent roots of the matrix A. Under such a transformation, $X_1, \cdots, X_n$ remain independent $N(0, 1)$. Let $F_n(t) = \Pr(Q_n \leqq t)$, then the problem is to find $F_n(t)$.

4. The solution.

Theorem. *Let $Q_n = \frac{1}{2}\sum_1^n a_i X_i^2$, where the X_i are independent $N(0, 1)$ variates and where $a_i > 0$ for $i = 1, 2, \cdots, n$. Let $Q_n^* = \frac{1}{2}\sum_1^n a_i^{-1} X_i^2$. Then*

(a) $$F_n(t) = \frac{t^{n/2}}{|A|^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)};$$

(b) *the series in* (a) *is absolutely convergent;*

(c) *for any two nonnegative integers r and s and every $t > 0$,*

$$S_{2s} = \sum_{k=0}^{2s} d_k > F_n(t) > \sum_{k=0}^{2r+1} d_k = S_{2r+1},$$

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[2] Part of a Ph.D. thesis presented to the Department of Mathematical Statistics, University of North Carolina.

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where

$$d_k = \frac{t^{n/2}}{|A|^{1/2}} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)}.$$

PROOF. Let $[R]$ represent the region where $\frac{1}{2}\sum_1^n a_i x_i^2 \leqq t$. Then

$$F_n(t) = (2\pi)^{-n/2} \int_{[R]} \cdots \int \exp\left[-\tfrac{1}{2} \sum_1^n x_i^2 \right] d\underset{\sim}{x}_n. \tag{1}$$

Expanding the exponential in the integrand, we get

$$F_n(t) = (2\pi)^{-n/2} \int_{[R]} \cdots \int \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\tfrac{1}{2} \sum_1^n x_i^2 \right)^k d\underset{\sim}{x}_n.$$

To evaluate integrals like $\int_{[R]} \cdots \int (\sum_1^n x_i^2)^k \, d\underset{\sim}{x}_n$, we expand the integrand according to the multinomial theorem and get

$$\sum_{i_1+\cdots+i_n=k} \frac{k!}{i_1! \cdots i_n!} \int_{[R]} \cdots \int \prod_{j=1}^n x_j^{2i_j} \, dx_j.$$

We shall make use of the Dirichlet integral

$$\int \cdots \int \prod_{j=1}^n x_j^{l_j - 1} \, d\underset{\sim}{x}_j = \prod_{j=1}^n \Gamma\left(\frac{l_j}{p_j}\right) c_j^{l_j} \Big/ \Gamma\left[\sum_1^n (l_j/p_j) + 1 \right], \quad -\infty < x_i < \infty,$$

where l_j, c_j, and p_j are all positive and the integration is over the region where $\sum_1^n (x_i/c_i)^{p_i} \leqq 1$. (See Edwards [1].) Putting

$$c_j = (2t/a_j)^{1/2}, \qquad p_j = 2, \qquad l_j = 2i_j + 1,$$

we find that

$$\int_{[R]} \cdots \int \left(\sum_1^n x_i^2 \right)^k d\underset{\sim}{x}_n = \frac{k!(2t)^{k+n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n + k + 1)} \sum_{i_1+\cdots+i_n=k} \frac{\Gamma(i_1 + \frac{1}{2}) \cdots \Gamma(i_n + \frac{1}{2})}{i_1! \cdots i_n! \, a_1^{i_1} \cdots a_n^{i_n}}.$$

The problem now is to evaluate this last expression. Recalling that if X_i is $N(0, 1)$, then $E(\frac{1}{2}X_i^2)^k = \Gamma(k + \frac{1}{2}) \,/\, \Gamma(\frac{1}{2})$, we find that

$$\begin{aligned} E(Q_n^*)^k &= E\left(\tfrac{1}{2} \sum_1^n a_i^{-1} X_i^2 \right)^k \\ &= 2^{-k} E \sum_{i_1+\cdots+i_n=k} \frac{k!}{i_1! \cdots i_n!} \frac{X_1^{2i_1} \cdots X_n^{2i_n}}{a_1^{i_1} \cdots a_n^{i_n}} \\ &= \pi^{-n/2} k! \sum_{i_1+\cdots+i_n=k} \frac{\Gamma(i_1 + \frac{1}{2}) \cdots \Gamma(i_n + \frac{1}{2})}{i_1! \cdots i_n! \, a_1^{i_1} \cdots a_n^{i_n}}. \end{aligned}$$

Consequently

$$(2) \qquad \int \cdots \int_{[R]} \left(\sum_1^n x_i^2\right)^k dx_n = \frac{\pi^{n/2}(2t)^{k+n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n + k + 1)} E(Q_n^*)^k,$$

$$F_n(t) = \frac{t^{n/2}}{|A|^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)}.$$

This proves part (a) of the theorem.

To show absolute convergence for part (b) we note that if $a = \min a_i$, then

$$Q_n^* \leqq \frac{1}{2a} \sum_1^n X_i^2, \qquad E(Q_n^*)^k \leqq \frac{\Gamma(\frac{1}{2}n + k)}{a^k\Gamma(\frac{1}{2}n)}.$$

Thus if $F_n^+(t)$ is the sum of the absolute values of the terms of the series for $F_n(t)$, then

$$F_n^+(t) \leqq \frac{t^{n/2}}{|A|^{1/2}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\Gamma(\frac{1}{2}n + k)}{a^k\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}n + k + 1)}$$

$$\leqq \frac{t^{n/2}}{|A|^{1/2}\Gamma(\frac{1}{2}n + 1)} e^{t/a} < \infty, \qquad |t| < \infty.$$

This proves part (b) of the theorem.

The bounds of part (c) are based on the fact that if r and s are any two non-negative integers, then, for $t > 0$ the Taylor expansion gives

$$\sum_{k=0}^{2s} \frac{(-t)^k}{k!} > e^{-t} > \sum_{k=0}^{2r+1} \frac{(-t)^k}{k!}.$$

Replacing t by $\sum x_i^2$ and using (1) and (2) then proves part (c) of the theorem.

REMARKS. (i) In case some of the latent roots are zero, that is the form is positive semidefinite of rank r, say, we need only replace n by r in the theorem and in the proof. (ii) The moments of Q_n^* are easy to obtain from the cumulants of Q_n^*. The rth cumulant of Q_n^* is $\frac{1}{2}(r-1)!\sum_1^n a_i^{-r}$ for $r = 1, 2, \cdots$. (iii) Clearly $S_0, S_2, S_4, \cdots$ is a sequence of upper bounds and $S_1, S_3, S_5, \cdots$ a sequence of lower bounds for $F_n(t)$. In practice we would compute a finite number of terms and then state that $\min S_{2s} > F_n(t) > \max S_{2r+1}$. The absolute value of the error thus committed is not greater than $\min S_{2s} - \max S_{2r+1}$, where r and s are non-negative integers.

5. Application to the distribution of a sum of squares in dependent variates. Let $X_1, \cdots, X_n$ have a joint multivariate normal distribution with zero means and inverse covariance matrix A. We wish to find the distribution of $R = \frac{1}{2}X'X$. Now

$$\Pr(R \leqq t) = \frac{|A|^{1/2}}{(2\pi)^{n/2}} \int \cdots \int_{[\frac{1}{2}x'x \leqq t]} \exp[-\tfrac{1}{2}x'Ax]\, dx_n.$$

We make an orthogonal transformation such that

$$\Pr(R \leqq t) = \frac{|A|^{1/2}}{(2\pi)^{n/2}} \int \cdots \int_{[\frac{1}{2}y'y \leqq t]} \exp\left[-\tfrac{1}{2}\sum_1^n a_i y_i^2\right] dy_n,$$

where $a_1, \cdots, a_n$ are the latent roots of the matrix A. We now let $z_i^2 = a_i y_i^2$, getting

$$\Pr(R \leqq t) = (2\pi)^{-n/2} \int \cdots \int_{\mathfrak{D}} \exp[-\tfrac{1}{2}z'z]\, dz_n, \qquad \mathfrak{D} = \left[\tfrac{1}{2}\sum_1^n a_i^{-1} z_i^2 \leqq t\right]$$

$$= \Pr(Q_n^* \leqq t).$$

Thus we can make use of the theorem.

REMARK. Combining the results of the theorem and the above application, it is easy to show that we could find the distribution of a definite quadratic form $X'AX$ where $X_1, \cdots, X_n$ have a multivariate normal distribution with covariance matrix B^{-1}, which distribution involves as parameters the latent roots of AB^{-1}.

We shall now state, without proof, an obvious corollary to the theorem, obtained by letting the first m_1 latent roots be a_1, the next m_2 latent roots be a_2, etc.

COROLLARY. *Let* $S_r = \frac{1}{2}(a_1\chi_{m_1}^2 + \cdots + a_r\chi_{m_r}^2)$, *where the* $\chi_{m_i}^2$, *for* $i = 1, \cdots, r$, *are independent random variables having a central chi square distribution with* m_i *degrees of freedom. Let*

$$S_r^* = \tfrac{1}{2}(a_1^{-1}\chi_{m_1}^2 + \cdots + a_r^{-1}\chi_{m_r}^2),$$

$$M = \sum_1^r m_i, \qquad a_i > 0, \qquad G_r(t) = \Pr(S_r \leqq t).$$

Then

(a) $$G_r(t) = \frac{t^{M/2}}{(a_1^{m_1} \cdots a_r^{m_r})^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{1}{2}M + k + 1)};$$

(b) *The series in* (a) *is absolutely convergent;*

(c) *For any two nonnegative integers* s *and* j *and every* $t > 0$,

$$\sum_{k=0}^{2s} d_k > G_r(t) > \sum_{k=0}^{2j+1} d_k, \qquad d_k = \frac{t^{M/2}}{(a_1^{m_1} \cdots a_r^{m_r})^{1/2}} \frac{(-t)^k}{k!} \frac{E(S_r^*)^k}{\Gamma(\frac{1}{2}M + k + 1)}.$$

REMARK. The moments of S_r^* are easy to compute from the cumulants. The jth cumulant of S_r^* is $\frac{1}{2}(j-1)!\sum_{i=1}^r m_i a_i^{-j}$.

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ON THE MAXIMUM NUMBER OF CONSTRAINTS OF AN ORTHOGONAL ARRAY[1]

By Esther Seiden

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1. Summary. R. C. Bose and K. A. Bush [1] showed how to make use of the maximum number of points, no three of which are collinear, in finite projective spaces for the construction of orthogonal arrays. In particular this enabled them to construct an orthogonal array (81, 10, 3, 3). They proved on the other hand that, in the case considered, the maximum number of constraints does not exceed 12 [1], (Theorem 2C). Hence they state: "We do not know whether we can get 11 or 12 constraints in any other way."

This paper shows first that a 10-rowed orthogonal array, constructed by the geometrical method, cannot be extended to an 11-rowed orthogonal array. It then shows that the number of constraints does not exceed 11. The problem of construction of an orthogonal array with 11 constraints remains unsolved.

This summary should serve, as well, as a correction to the statement made in the abstract, "A remark on the geometrical method of construction of an orthogonal array," published in the *Annals of Mathematical Statistics*, Vol. 25 (1954), p. 177–178, which claimed the nonexistence of an orthogonal array of 11 constraints also.

2. Introduction. The proof is based on an algebraic property of orthogonal arrays, pointed out by Bose and Bush [1]. Let n_{ij}^k denote the number of columns belonging to an array consisting of k rows that have j coincidences (j elements equal) with the ith column. A necessary condition for an array $(\lambda s^t, k, s, t)$ to be orthogonal is that whatever be the number h such that $0 \leqq h \leqq t$, the following equalities hold

$$\sum_{j=0}^{k} n_{ij}^k C_h^j = C_h^k (\lambda s^{t-h} - 1), \qquad i = 1, 2, \cdots, \lambda s^t$$

where the C's are binomial coefficients.

In the case considered, the equalities become, for $i = 1, 2, \cdots, 81$,

$$(1) \qquad \begin{aligned} &\sum_{j=0}^{k} n_{ij}^k = 80 \qquad && \sum_{j=0}^{k} j(j-1)\, n_{ij}^k = 8k(k-1) \\ &\sum_{j=0}^{k} j\, n_{ij}^k = 26k \qquad && \sum_{j=0}^{k} j(j-1)(j-2)\, n_{ij}^k = 2k(k-1)(k-2). \end{aligned}$$

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Now at Howard University.

Equations (1) will be used throughout the proof in the special case of $i = 1$. Moreover, it will be assumed, without loss of generality, that the first column consists of zeros only.

3. Derivations.

LEMMA 1. *An orthogonal array* (81, 10, 3, 3), *with* 10 *constraints and* $n_{1j}^{10} = 0$ *for* $j \geqq 5$ *cannot be extended to an* 11*-rowed orthogonal array.*

PROOF. The third and fourth equations of (1) becomes in this case

$$2n_{12}^{10} + 6n_{13}^{10} = 720 - 12n_{14}^{10}$$

$$6n_{13}^{10} = 1440 - 24n_{14}^{10}.$$

Since $n_{13}^{10} \geqq 0$ and $n_{12}^{10} \geqq 0$, both equations imply $n_{14}^{10} \leqq 60$. On the other hand, subtracting the second equation from the first gives $n_{14}^{10} \geqq 60$. Consequently $n_{14}^{10} = 60$ and $n_{12}^{10} = n_{13}^{10} = 0$. Then the second and first equations of (1) imply $n_{11}^{10} = 20$ and $n_{10}^{10} = 0$, respectively. It remains to show now that an array satisfying this solution cannot be extended to an 11-rowed orthogonal array. We observe first that this solution implies $n_{10}^{11} = n_{13}^{11} = 0$ and thus all other n's are uniquely determined, namely $n_{15}^{11} = 9$, $n_{14}^{11} = 60$, $n_{12}^{11} = -10$, $n_{11}^{11} = 21$, which is clearly impossible.

THEOREM 1. *A* 10*-rowed orthogonal array constructed by the method employed by Bose and Bush* [1] *cannot be extended to an* 11*-rowed orthogonal array.*

PROOF. Since the maximum number of points of which no three are collinear in the projective plane considered is equal to four, it is easy to see that the method of construction cannot lead to a column having more than four coincidences with the first column. Hence the theorem is an immediate consequence of Lemma 1.

LEMMA 2. *The number of coincidences between any two columns of a* 6*-rowed orthogonal array is less than six, provided that* $\lambda = s = t = 3$.

PROOF. Consider equations (1) for $k = 6$ and $i = 1$. Express n_{10}^{6}, n_{11}^{6}, n_{12}^{6}, and n_{13}^{6} in terms of n_{14}^{6}, n_{15}^{6}, and n_{16}^{6}. The equations become

$$n_{10}^{6} = 4 + n_{14}^{6} + 4n_{15}^{6} + 10n_{16}^{6}, \qquad n_{11}^{6} = 36 - 4n_{14}^{6} - 15n_{15}^{6} - 36n_{16}^{6},$$

$$n_{12}^{6} = 6n_{14}^{6} + 20n_{15}^{6} + 45n_{16}^{6}, \qquad n_{13}^{6} = 40 - 4n_{14}^{6} - 10n_{15}^{6} - 20n_{16}^{6}.$$

If $n_{16}^{6} > 0$, all the n's are uniquely determined, namely

$$n_{16}^{6} = 1, \quad n_{15}^{6} = n_{14}^{6} = 0, \quad n_{13}^{6} = 20, \quad n_{12}^{6} = 45, \quad n_{11}^{6} = 0, \quad n_{10}^{6} = 14.$$

This means that every 4-rowed subarray must satisfy the equality $n_{14}^{4} = 1$, and again all n's will be uniquely determined as $n_{13}^{4} = 4$, $n_{12}^{4} = 30$, $n_{11}^{4} = 28$, and $n_{10}^{4} = 17$. Hence, if we delete from the original array any two rows, the number of columns which have no coincidences with the first column will increase by three. Since $n_{11}^{6} = 0$, every pair of rows will contain exactly six zeros, contained in three columns each having two zeros. Moreover, different pairs must contain zeros belonging to different such columns. The number of pairs of rows is equal to

20. Hence n_{12}^{6} would have to be equal to 60, contrary to the assumption that $n_{12}^{6} = 45$. This proves Lemma 2.

COROLLARY. *Any orthogonal array* (81, k, 3, 3) *satisfies the equalities* $n_{ij}^{k} = 0$, *provided that* $j \geqq 6$ *and* $1 \leqq i \leqq 81$.

LEMMA 3. *An orthogonal array with* 11 *constraints satisfies one of the nine solutions*

Solution	n_{ij}					
	n_{15}^{11}	n_{14}^{11}	n_{13}^{11}	n_{12}^{11}	n_{11}^{11}	n_{10}^{11}
I	11	55	0	0	11	3
II	12	52	2	2	8	4
III	13	49	4	4	5	5
IV	13	50	0	10	1	6
V	14	45	0	10	6	5
VI	14	46	6	6	2	6
VII	15	42	12	2	3	6
VIII	16	39	14	4	0	7
IX	17	35	20	0	1	7

PROOF. In view of Lemma 2, equations (1) imply

$$2n_{12}^{11} + 3n_{13}^{11} = -110 + 10n_{15}^{11}, \qquad 3n_{11}^{11} + 2n_{12}^{11} = 88 - 5n_{15}^{11}.$$

Hence, $11 \leqq n_{15}^{11} \leqq 17$, since all the n's are nonnegative. All solutions which could be satisfied by an orthogonal array with 11 constraints can now be easily enumerated. For instance, we may express n_{10}^{11}, n_{11}^{11}, n_{12}^{11}, and n_{13}^{11} in terms of n_{14}^{11} and n_{15}^{11}. Then

$$n_{10}^{11} = -96 + n_{14}^{11} + 4n_{15}^{11}, \qquad n_{11}^{11} = 396 - 4n_{14}^{11} - 15n_{15}^{11},$$

$$n_{12}^{11} = -550 + 6n_{14}^{11} + 20n_{15}^{11}, \qquad n_{13}^{11} = 330 - 4n_{14}^{11} - 10n_{15}^{11}.$$

Consider now, for example, $n_{15}^{11} = 11$. By the last equality, $n_{14}^{11} \leqq 55$, and by the third equality, $n_{14}^{11} \geqq 55$. Hence $n_{14}^{11} = 55$ and all other n's are uniquely determined.

LEMMA 4. *If an orthogonal array of* 12 *constraints exists, it must satisfy one of the two solutions*

Solution	n_{ij}					
	n_{15}^{12}	n_{14}^{12}	n_{13}^{12}	n_{12}^{12}	n_{11}^{12}	n_{10}^{12}
I′	27	42	2	0	3	6
II′	28	39	4	2	0	7

PROOF. Equations (1) imply for $k = 12$

$$3n_{13}^{12} + 2n_{12}^{12} = -264 + 10n_{15}^{12}, \qquad 3n_{11}^{12} + 2n_{12}^{12} = 144 - 5n_{15}^{12}.$$

Thus $27 \leqq n_{15}^{12} \leqq 28$. The same method as before shows that I′ and II′ are the only solutions of equations (1) for $k = 12$ and $n_{15}^{12} = 27$ or $n_{15}^{12} = 28$.

THEOREM 2. *The number of constraints of an orthogonal array does not exceed* 11, *provided that* $\lambda = s = t = 3$.

PROOF. It is enough to prove that neither Solution I′ nor II′ can be satisfied by an orthogonal array. As to solution I′, since $n_{10}^{12} = 6$, every 11-rowed subarray would have to satisfy the inequality $n_{10}^{11} \geqq 6$. Hence it remains to consider solutions IV, VI, VII, VIII and IX. It is easy to rule out all except VII since the rest could lead to 12-rowed arrays for which, if $n_{10}^{12} = 6$, n_{11}^{12} could be at most equal to two, contrary to the assumption that it is three. Hence, every 11-rowed subarray of an array satisfying I′ would have to satisfy VII. This is impossible, since $n_{11}^{12} \neq 0$. Thus, if we delete from the array a row which contains a zero belonging to a column having just one zero, we must get an 11-rowed subarray for which $n_{10}^{11} \geqq 7$.

The proof regarding solution II′ is analogous. Here every 11-rowed subarray would have to satisfy either VIII or IX. Solution IX is impossible, since it could lead to a 12-rowed array for which, if $n_{12}^{10} = 7$, n_{12}^{12} is at most equal to one. As before, it is impossible that every 11-rowed subarray of an array satisfying II′ will satisfy IX, because deletion from this array of a row containing a zero belonging to a column which has two zeros only must yield an 11-rowed subarray for which $n_{11}^{11} > 0$. This completes the proof of Theorem 2.

It is easy to show that no 11-rowed orthogonal arrays exist which satisfy solutions II, III, or V. As to the remaining solutions, the problem is unsolved.

I wish to thank R. C. Bose for making accessible to me his unpublished work on this subject.

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NOTES

ON A SEQUENTIAL TEST FOR THE GENERAL LINEAR HYPOTHESIS[1]

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1. Introduction. A few years ago I reported [1] on a sequential method for testing the general linear hypothesis, but held up publication until some of the properties of the method had been investigated further. Johnson [2] recently published a paper in which he obtained the same sequential test, but from an entirely different point of view. His method of derivation is based on showing, by means of a theorem of Cox [3], that the likelihood ratio approach to the problem can be used successfully. My method is a direct generalization of Wald's sequential t-test. This note outlines the nature of this generalization, and also points out some additional properties of the test.

2. Method. The general linear hypothesis assumes that the variables x_1, x_2, $\cdots$, x_l with means μ_1, μ_2, $\cdots$, μ_l and common variance σ^2 possess the frequency function

$$(1) \qquad f(x_1, \cdots, x_l) = (2\pi\sigma)^{-l} \exp\left\{\frac{-1}{2\sigma^2}\left[\sum_{i=1}^{k}(x_i - \mu_i)^2 + \sum_{i=k+1}^{l} x_i^2\right]\right\}.$$

It tests the hypothesis H_0: $\mu_1 = \cdots = \mu_p = 0$, for $p \leqq k$. The means μ_{k+1}, $\cdots$, μ_l have the value zero. The parameters μ_{p+1}, $\cdots$, μ_k, and σ are nuisance parameters, and therefore make H_0 a composite hypothesis.

Following Wald's [4] procedure and notation for the sequential t-test, let the parameter space Ω be divided into the three regions ω_a, ω_r, and $\Omega - \omega_a - \omega_r$. The region ω_a will be chosen as that part of Ω where H_0 holds. The region ω_r will be chosen as that part of Ω where

$$\sum_{i=1}^{p} \frac{\mu_i^2}{\sigma^2} \geqq \lambda_0$$

where λ_0 is a selected constant. The boundary of ω_r will be denoted by S_r. As normalized weight functions choose

$$v_a(\theta) = \begin{cases} a_1 \sigma^{b_1} (2c)^{p-k}, & 0 \leqq \sigma \leqq c, |\mu_i| \leqq c, i = p+1, \cdots, k, \\ 0, & \text{elsewhere;} \end{cases}$$

$$v_r(\theta) = \begin{cases} a_2 \sigma^{b_2} (2c)^{p-k}, & 0 \leqq \sigma \leqq c, |\mu_i| \leqq c, i = p+1, \cdots, k, \\ 0, & \text{elsewhere.} \end{cases}$$

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Here a_1, a_2, b_1, and b_2 are certain related constants to be specified later. With these weight functions and with $\prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{l\alpha})$ denoting the likelihood function for a sample of size n from (1), form the quantities

$$p_{1nc} = \int_{S_{rc}} v_r(\theta) \prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{l\alpha})\, dS_{rc},$$

$$p_{0nc} = \int_{\omega_{ac}} v_a(\theta) \prod_{\alpha=1}^{n} f(x_{1\alpha}, \cdots, x_{l\alpha})\, d\theta.$$

Here S_{rc} is that part of the surface S_r and ω_{ac} is that part of the region ω_a where $0 \leqq \sigma \leqq c$ and $|\mu_i| \leqq c$ for $i = p + 1, \cdots, k$. These are truncations of S_r and ω_a to permit the existence of the necessary integrals.

Now choose $b_2 = b_1 + 1 - p$, form the ratio p_{1nc} / p_{0nc}, and allow c to become infinite. The resulting limit, which will be denoted by p_{1n} / p_{0n}, is then treated as the likehood ratio in the standard sequential probability ratio test. By choosing $b_1 = -1$ and $l = k$ it can be shown that this test reduces to the test proposed by Johnson.

The procedure, in outline, for obtaining the reduced form of this test is as follows. The numerator integral in p_{1n} / p_{0n}, which is a surface integral over S_r, is expressed as an integral over the $\mu_1, \cdots, \mu_p$ plane, and the denominator integral is expressed as an integral with respect to σ. Then, letting

$$z_{i\alpha} = \begin{cases} x_{i\alpha}, & i = 1, \cdots, p, k+1, \cdots, l, \\ x_{i\alpha} - \bar{x}_i, & i = p+1, \cdots, k, \end{cases}$$

it is shown that p_{1n} / p_{0n} is a homogeneous function of degree zero in the $z_{i\alpha}$. This permits the replacing of $z_{i\alpha}$ by $z_{i\alpha} / \sqrt{\sum\sum z_{i\alpha}^2}$, which in turn enables one to evaluate the denominator integral. The numerator integral is evaluated by introducing spherical coordinates, expanding one of the exponential functions in the integral, and integrating term by term. The resulting series will be found to reduce to a confluent hypergeometric function. The result of these manipulations is the expression

$$\frac{p_{1n}}{p_{0n}} = e^{-n\lambda_0/2} F(\tfrac{1}{2}[nl + p - k - b_1 - 1], \tfrac{1}{2}p, \delta^2/2\lambda_0) \tag{2}$$

where $\delta^2 = n\lambda_0^2\, n \sum_{i=1}^{p} \bar{x}_i^2 / [\sum_{\alpha=1}^{n} \sum_{i=1}^{l} x_{i\alpha}^2 - n \sum_{i=p+1}^{k} \bar{x}_i^2]$.

This derivation requires that $b_1 \leqq 0$. If b_1 is chosen equal to -1, the preceding test reduces to the Johnson test except for one minor feature. Johnson's formulation of the general linear hypothesis does not include the variables $x_{k+1}, \cdots, x_l$, and as a result our first parameter values in the confluent hypergeometric function are not in agreement unless $l = k$. His derivation applies equally well to the formulation given in (1), and then his test will be identical with (2) for $b_1 = -1$. The choice of $b_1 = -1$ in (2) is probably as good a choice as any other.

3. Properties. If the principle of invariance is applied, the sequential test (2) will possess an interesting property.

From (1) the frequency function for a sample of size n is

$$(3) \qquad L = (2\pi\sigma)^{-nl} \exp\left\{\frac{-1}{2\sigma^2}\sum_{\alpha=1}^{n}\left[\sum_{i=1}^{k}(x_{i\alpha}-\mu_i)^2 + \sum_{i=k+1}^{l} x_{i\alpha}^2\right]\right\}.$$

By an orthogonal transformation for which $x'_{i1} = (x_{i1} + \cdots + x_{in}) / \sqrt{n}$, for $i = 1, \cdots, k$, (3) can be reduced to the canonical form

$$(4) \quad L' = (2\pi\sigma)^{-nl} \exp\left\{\frac{-1}{2\sigma^2}\left[\sum_{\alpha=2}^{n}\sum_{i=1}^{k} x'^2_{i\alpha} + \sum_{\alpha=1}^{n}\sum_{i=k+1}^{l} x'^2_{i\alpha} + \sum_{i=1}^{k}(x'_{i1} - \sqrt{n}\,\mu_i)^2\right]\right\}.$$

From this it follows that the problem specification will remain invariant under the following groups of transformations:

(i) $\bar{x}_{i1} = x'_{i1} + c, \quad \text{for } i = p+1, \cdots, k;$

(ii) any orthogonal transformation on the variables $x'_{i\alpha}$ when the variables $x'_{11}, \cdots, x'_{k1}$ are deleted, and any orthogonal transformation on the variables $x'_{11}, \cdots, x'_{p1}$;

(iii) $\bar{x}_{i\alpha} = cx_{i\alpha}, \quad \text{for } i = 1, \cdots, l, \text{ and } \alpha = 1, \cdots, n.$

Standard methods will show that the maximal invariant under these groups of transformations is

$$\phi_n = \sum_{i=1}^{p} x'^2_{i1} \Big/ \left[\sum_{\alpha=2}^{n}\sum_{i=1}^{k} x'^2_{i\alpha} + \sum_{\alpha=1}^{n}\sum_{i=k+1}^{l} x'^2_{i\alpha}\right].$$

In terms of the original variables $x_{i\alpha}$, the expression for ϕ_n reduces to

$$\phi_n = n\sum_{i=1}^{p} \bar{x}_i^2 \Big/ \left[\sum_{\alpha=1}^{n}\sum_{i=1}^{l} x_{i\alpha}^2 - n\sum_{i=p+1}^{k} \bar{x}_i^2\right].$$

Thus, if the test of H_0 is to be an invariant sequential test, it must be based on the sequence of variables $\phi_1, \phi_2, \phi_3, \cdots$. Since (2) is based on this sequence, it is an invariant test. The distribution of ϕ_n depends on the parameters through the single parameter

$$\lambda = \sum_{i=1}^{p} \frac{\mu_i^2}{\sigma^2}.$$

As a consequence, for any invariant test, testing the hypotheses

$$H_0 : \mu_i = 0 \qquad \text{vs.} \qquad H_1 : \mu_i = \mu'_i, \qquad i = 1, \cdots, p,$$

is equivalent to testing the hypothesis $\lambda = 0$ against the alternative $\lambda = \lambda_0$ where $\lambda_0 = \sum_1^p \mu_i'^2 / \sigma^2$.

Johnson has shown that test (2) is the probability ratio test for testing $\lambda = 0$ against $\lambda = \lambda_0$ based on the variables $\phi_1, \phi_2, \phi_3, \cdots$. If the latter variables were independent, it would follow from the optimum property of the probability ratio test that test (2) with $b_1 = -1$ is the optimum invariant test of H_0. However, since these variables are not independent, this conclusion does not

follow. It may be that the use of decision limits which depend upon the value of n will produce a better test than (2), which uses the customary constant limits for the sequential probability ratio test.

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TWO COMMENTS ON "SUFFICIENCY AND STATISTICAL DECISION FUNCTIONS"

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In the following comments we employ the notation and definitions of [1]. The first comment answers a question raised in [1] by giving an example of a necessary and sufficient subfield which cannot be induced by a statistic. The second remark clarifies this example somewhat by discussing the connection between statistics and subfields in general. It was hoped that this connection would be so close as to provide the answer to another question raised in [1]: whether the existence of a necessary and sufficient subfield implies that of a necessary and sufficient statistic. However, an example given at the end of the second comment shows that such a result cannot be proved without making deeper use of sufficiency.

1. A counter example. The following result was communicated to us by David Blackwell.

LEMMA 1. (*Blackwell*). *Let* $\mathbf{S}_0$ *be a proper subfield of* $\mathbf{S}$ *and suppose that for each* x *the set* $\{x\}$ *consisting of the single point* x *is in* $\mathbf{S}_0$. *Then* $\mathbf{S}_0$ *cannot be induced by a statistic.*

PROOF. Suppose there exists such a statistic, say T, and let $\mathbf{T}$ be the field of sets B in the range of T such that $T^{-1}(B) \varepsilon \mathbf{S}$. Since $\{x\} \varepsilon \mathbf{S}_0$, there exists $B \varepsilon \mathbf{T}$ such that $T^{-1}(B) = \{x\}$, and, by definition of $\mathbf{T}$, a set $A \varepsilon \mathbf{S}$ such that $T(A) = B$. We therefore have $T^{-1}[T(A)] = \{x\}$, and since always $T^{-1}[T(A)] \supseteq A$, we have that $T^{-1}[T(x)] = x$ for all x. Therefore, if A is any set in $\mathbf{S}$, we see that $T^{-1}[T(A)] = A$ so that $A \varepsilon \mathbf{S}_0$ and hence our assumption that $\mathbf{S}_0$ is induced by T implies that $\mathbf{S}_0 = \mathbf{S}$.

We now give an example of a necessary and sufficient subfield that cannot be

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induced by a statistic. A trivial such example would be the case of two normal distributions defined over the Lebesgue sets $\mathbf{S}$ of the real line. Then $\mathbf{S}_0$, the class of Borel sets, is a necessary and sufficient subfield, and this cannot, by Lemma 1, be induced by a statistic. The example is, however, uninteresting since $\mathbf{S}_0 = \mathbf{S}[\mathbf{S}, P]$ and $\mathbf{S}$ is induced by the necessary and sufficient statistic $T(x) = x$. In this case, therefore, the necessary and sufficient statistic is equivalent to the necessary and sufficient subfield.

An example in which this is not the case can be based on a problem discussed by Halmos ([2], p. 71, prob. 2). Let X be the interval $(0, 1)$, let $\mathbf{S}_0$ be the class of Lebesgue sets of X, and let p denote Lebesgue measure on $\mathbf{S}_0$. Let $M \subset X$ be a fixed set which is not Lebesgue-measurable, and let the complement of M be denoted by $\bar{M}$. Define $\mathbf{S}$ to be the class of all sets of the form $EM + F\bar{M}$, with E and F in $\mathbf{S}_0$. Then $\mathbf{S}$ is a field containing $\mathbf{S}_0$. We take $(X, \mathbf{S})$ to be the sample space. To define P, let G and H be fixed sets in $\mathbf{S}_0$ such that

$$G \subset M \subset H \text{ and } p(G) = p_*(M),\ p^*(M) = p(H),$$

where p_* and p^* denote inner and outer Lebesgue measure. Let $D = H - G$, and for any set $N = EM + F\bar{M}$ in $\mathbf{S}$ define

$$\lambda(N) = p[(EM + F\bar{M})\bar{D}] + \tfrac{1}{2}p(ED) + \tfrac{1}{2}p(FD).$$

Halmos shows that λ is a measure on $\mathbf{S}$ such that $\lambda = p$ on $\mathbf{S}_0$. Let the set P of probability measures on $\mathbf{S}$ consist of the two measures λ and μ, where μ is defined by $d\mu = 2x\, d\lambda$.

It is easily seen that in this case (i) $\mathbf{S}_0$ is a necessary and sufficient subfield (cf. Theorem 6.2 of [1]), (ii) $T_0(x) = x$ is a necessary and sufficient statistic (cf. Theorem 6.3 of [3]), and (iii) P is a completed set of measures on $\mathbf{S}$, that is, if a set A is $\mathbf{S}$-P-null, then every $B \subset A$ is in $\mathbf{S}$ (and therefore $\mathbf{S}$-P-null). We shall show that no necessary and sufficient subfield is equivalent to a statistic; a fortiori, no such subfield is inducible by a statistic.

Suppose to the contrary that $\mathbf{S}_*$ is a necessary and sufficient subfield, and that $\mathbf{S}_* = \mathbf{S}_T[\mathbf{S}, P]$, where $\mathbf{S}_T$ is the subfield induced by a statistic T. It follows from the essential uniqueness of the necessary and sufficient subfield (cf. Corollary 6.2 of [1]) that $\mathbf{S}_T$ is necessary and sufficient. Consequently, by (i), $\mathbf{S}_T$ is equivalent to $\mathbf{S}_0$. The sufficiency of $\mathbf{S}_T$ means that T is a sufficient statistic. Hence, by (ii), T_0 is essentially a function of T. More precisely, there exists a function F on the range of T into X, and an $\mathbf{S}$-P-null set N, such that $T_0(x) = F(T(x))$ on $X - N$. This, together with (iii), implies that the subfield induced by T_0 is essentially a subfield of $\mathbf{S}_T$. However, T_0 induces $\mathbf{S}$ itself, so that $\mathbf{S}_T$ must be equivalent to $\mathbf{S}$.

Thus $\mathbf{S}_T$ is equivalent to $\mathbf{S}_0$ and also to $\mathbf{S}$. We conclude that $\mathbf{S}_0$ is equivalent to $\mathbf{S}$, that is, $\mathbf{S}_0 = \mathbf{S}[\mathbf{S}, P]$. Since $\bar{M}$ is in $\mathbf{S}$, and P contains λ, this conclusion implies that there exists a set in $\mathbf{S}_0$, E_0 say, such that the symmetric difference of E_0 and $\bar{M}$ is of λ-measure zero. This is, however, a contradiction, since the symmetric difference in question is $E_0M + \bar{E}_0\bar{M}$, and its λ-measure is not less than $\frac{1}{2}p(E_0D) + \frac{1}{2}p(\bar{E}_0D) = \frac{1}{2}p(D)$, and $p(D) = p^*(M) - p_*(M) > 0$.

2. The connection between subfields and statistics. The above Lemma of Blackwell provides a necessary condition for a subfield to be inducible by a statistic. We shall now obtain a necessary and sufficient condition. As is pointed out in [1], any subfield $\mathbf{S}_0$ of $\mathbf{S}$ induces a partition π if we put $x \sim x'$, provided for all $A_0 \in \mathbf{S}_0$ we have $x \in A_0 \Leftrightarrow x' \in A_0$. Let E_x denote the set of π containing x. Then we may characterize E_x as the largest set containing x and such that for all $A_0 \in \mathbf{S}_0$ either $E_x \subseteq A_0$ or $E_x \subseteq \bar{A}_0$.

Not every partition π can be induced by a subfield of $\mathbf{S}$, and if it can there may be more than one subfield inducing it. Let us denote by C_π the (possibly empty) class of all subfields of $\mathbf{S}$ that induce π. We then have

LEMMA 2. *If C_π is nonempty, it contains a largest member $\mathbf{S}_\pi$ which is given by*

$$\mathbf{S}_\pi = \{A : A \in \mathbf{S}, x \in A \Rightarrow E_x \subseteq \mathrm{A} \text{ for all } x \in X\}.$$

PROOF. Since C_π is nonempty, there exists a subfield $\mathbf{S}_1$ of $\mathbf{S}$ which induces π. For any $A_1 \in \mathbf{S}_1$ and any $x \in X$ it follows that $x \in A_1 \Rightarrow E_x \subseteq A_1$ so that $\mathbf{S}_1 \subseteq \mathbf{S}_\pi$. Therefore, if π' is the partition induced by $\mathbf{S}_\pi$, we see that π' is a refinement of π. (In general, if $\mathbf{S}_i$ induces π_i for $i = 0$ or 1 and if $\mathbf{S}_1 \subseteq \mathbf{S}_0$, then π_0 is a refinement of π_1).

We shall now prove that, conversely, π is a refinement of π'. This will show that $\pi' = \pi$ and hence that $\mathbf{S}_\pi$ induces π. Since we have already shown that $\mathbf{S}_\pi$ contains any $\mathbf{S}_1$ that induces π, this will establish that $\mathbf{S}_\pi$ is the largest member of C_π.

Let E'_x be the set of π' containing x. Then E'_x is the intersection of all $A \in \mathbf{S}_\pi$ that contain x. Since by definition of $\mathbf{S}_\pi$ all of these sets also contain E_x it follows that $E_x \subseteq E'_x$, as was to be proved.

We can now state

LEMMA 3. *Let C_π be nonempty. Then one and only one of the subfields constituting C_π, can be induced by a statistic, namely $\mathbf{S}_\pi$.*

Hence: A necessary and sufficient condition for a subfield $\mathbf{S}_0$ to be inducible by a statistic is that, $\mathbf{S}_0 = \mathbf{S}_\pi$, if $\mathbf{S}_0$ induces π.

PROOF. We remark first that if T is a statistic, then the subfield $\mathbf{S}_0$ induced by it is the class of all $A_0 \in \mathbf{S}$ for which $T^{-1}[T(A)] = A$. Now let E_x be defined relative to π as before and let $T(x) = E_x$. Then $T^{-1}[T(x)] = E_x$ and we see that $A_0 \in \mathbf{S}_0$ (the subfield induced by T) if and only if $A_0 \in \mathbf{S}$ and $x \in A_0 \Rightarrow E_x \subseteq A_0$. This shows that $\mathbf{S}_0 = \mathbf{S}_\pi$ and hence that $\mathbf{S}_\pi$ can be induced by a statistic.

On the other hand, let T be any statistic whose subfield $\mathbf{S}_0$ induces π and let $F_x = \{x' : T(x') = T(x)\}$. Then

$$x \in A_0 \Rightarrow F_x \subseteq A_0, \qquad x \in \bar{A}_0 \Rightarrow F_x \subseteq \bar{A}_0.$$

It follows from the characterization of E_x given earlier that $F_x \subseteq E_x$. Therefore, $A \in \mathbf{S}_\pi, x \in A \Rightarrow F_x \subseteq \mathrm{A}$, and hence $A \in \mathbf{S}_0$. It follows that $\mathbf{S}_0 = \mathbf{S}_\pi$.

One might hope that Lemma 3 would establish the existence of a necessary and sufficient statistic as $\mathbf{S}_\pi$, where π is the partition induced by a necessary and sufficient subfield. Unfortunately, however, the notion of statistic is not invariant

under equivalence [$\mathbf{S}$, P]. A subfield equivalent to a statistic need not itself be a statistic. In an attempt to avoid this difficulty, one may define a *pseudo-statistic* as any subfield equivalent to a statistic. If Lemma 3 remained valid for pseudo-statistics in the sense that a member of C_τ is a pseudo-statistic if and only if it is equivalent to $\mathbf{S}_\tau$, this would establish the desired result.

The following example shows that this stronger version of Lemma 3 is not correct. Let $\mathbf{S}_\tau$ be the class of all Lebesgue sets on the real line and $\mathbf{S}_0$ the class of all Lebesgue sets differing only by a set 0 from a set symmetric with respect to the origin. Clearly, $\{x\} \in \mathbf{S}_0$ for all x so that $\mathbf{S}_0 \in C_\tau$. Also $\mathbf{S}_0$ is a pseudo-statistic since it is equivalent to the subfield induced by $T(x) = |x|$. But clearly $\mathbf{S}_0$ and $\mathbf{S}_\tau$ are not equivalent.

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A NOTE ON CONFIDENCE SETS FOR RANDOM VARIABLES

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Suppose the chance variables $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ have a joint probability distribution depending on the unknown parameters $\theta_1, \cdots, \theta_k$, but otherwise of known form. We assume that there is a set of sufficient statistics for

$\theta_1, \cdots, \theta_k$, denoted by $T_1(X_1, \cdots, X_m, Y_1, \cdots, Y_n), \cdots,$

$T_r(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$. We shall let X denote the vector $(X_1, \cdots, X_m)$, Y the vector $(Y_1, \cdots, Y_n)$, θ the vector $(\theta_1, \cdots, \theta_k)$, and $T(X, Y)$ the vector $(T_1(X, Y), \cdots, T_r(X, Y))$. $P_\theta(A)$ shall denote the probability of A when the vector of parameters equals θ, and $P_\theta(A \mid B)$ shall denote the conditional probability of A given B when the vector of parameters equals θ.

Given a number α between 0 and 1, if for each vector X we can find a set $S(X)$ in n-dimensional Euclidean space such that $P_\theta(Y \text{ in } S(X)) = \alpha$ identically in θ, then the system of sets $S(X)$ is called a "parameter-free confidence set of level α for the random vector Y."

Since $T(X, Y)$ is a set of sufficient statistics for θ, the joint conditional distribution of Y given that $T(X, Y) = t = (t_1, \cdots, t_r)$ is independent of θ. But then for any given vector t, it is possible to construct a region $S'(t)$ in n-dimensional Euclidean space such that $P_\theta(Y \text{ in } S'(t) \mid T(X, Y) = t) = \alpha$ identically in θ

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(This last sentence is true if the conditional distribution of Y given that $T(X, Y) = t$ is absolutely continuous. In the other cases only approximate equality may be possible.) Then we have $P_\theta[Y \text{ in } S'(T(X, Y))] = \alpha$ identically in θ. Now for any X, we define $S(X)$ as the set of vectors Y such that Y is in $S'(T(X, Y))$. Clearly, $S(X)$ is a parameter-free confidence set of level α.

This construction is the exact analogue of Neyman's construction of confidence sets for a parameter, as described in [1].

As an example, we discuss the problem that motivated this note. We have two solutions, each consisting of a certain type of particles suspended in water. The relative concentration of the two solutions is known, but not the absolute concentration of either. That is, we do not know the average number of particles per unit volume for either solution, but we do know the ratio of the average numbers of particles per unit volume for the two solutions. In taking this ratio, we shall understand that the numerator refers to the second solution, and shall denote the ratio by R. V_1 cubic centimeters of the first solution are drawn and the number of particles in this volume is counted. We denote this number by X. V_2 cubic centimeters from the second solution are to be injected into an experimental animal. We denote the number of particles injected into the animal by Y. Y will never be directly observed. Given a number α between 0 and 1, the problem is to find two functions of X, say $L_1(X)$ and $L_2(X)$, with $L_1(X) < L_2(X)$, such that $P[L_1(X) \leqq Y \leqq L_2(X)] = \alpha$ no matter what the values of the unknown absolute concentrations are. From familiar considerations, it is reasonable to assume that X and Y are independently distributed, each with a Poisson distribution. If θ denotes the expected value of X, then the expected value of Y is $r\theta$, where $r = R(V_2/V_1)$ and is therefore known. θ is unknown. Then we wish to have $P_\theta[L_1(X) \leqq Y \leqq L_2(X)] \equiv \alpha$ identically in θ. It is easily verified that $X + Y$ is a sufficient statistic for θ, and that

$$P(Y = y | X + Y = z) = \binom{z}{y}[r/(1 + r)]^y[1/(1 + r)]^{z-y},$$

that is, the conditional distribution of Y given $X + Y$ is binomial and is independent of θ. For any given value of $X + Y$, we can find two numbers $M_1(X + Y)$ $M_2(X + Y)$ so that $P[M_1(X + Y) \leqq Y \leqq M_2(X + Y) \mid X + Y]$ is approximately equal to α irrespective of the value of θ. If X is large enough to make both $Xr/(1 + r)$ and $X/(1 + r)$ large, then the conditional distribution of Y given $X + Y$ is nearly normal, and we have as a good approximation

$$M_1(X + Y) = (X + Y)r / (1 + r) - k\sqrt{(X + Y)r/(1 + r)^2},$$

$$M_2(X + Y) = (X + Y)r / (1 + r) + k\sqrt{(X + Y)r/(1 + r)^2},$$

where k depends on α and is found from the table of the normal distribution. Then the confidence set $S(X)$ is the set of all values of Y such that $M_1(X + Y) \leqq Y \leqq M_2(X + Y)$, and using the approximate values of $M_1(X + Y)$ and $M_2(X + Y)$ from the preceding sentence, we find that $S(X)$ consists of all

values of Y between the limits

$$L_1(X) = \tfrac{1}{2}\{2rX + rk^2 - k\sqrt{k^2r^2 + 4rX(1 + r)}\}, \quad \text{and}$$

$$L_2(X) = \tfrac{1}{2}\{2rX + rk^2 + k\sqrt{k^2r^2 + 4rX(1 + r)}\}.$$

Therefore, for these limits, we have that $P[L_1(X) \leqq Y \leqq L_2(X)]$ is approximately equal to α, no matter what the value of θ is.

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AN INCONSISTENCY OF THE METHOD OF MAXIMUM LIKELIHOOD

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An example was given by Neyman and Scott [2] to show that there are situations where the method of maximum likelihood leads to inconsistent estimators. In their example considered, the observations were supposed to be drawn from an infinite sequence of distinct populations involving an infinite sequence of nuisance parameters.

An example is given here to demonstrate that even in simple situations where all the observations are independently and identically distributed and involve only one unknown parameter, the method of maximum likelihood may lead us astray. The example typifies the situations where the correct method of setting up a point estimate should begin with a test of hypothesis between two composite alternatives.

Let A be the set of all rational numbers in the closed interval (0, 1) and B any countable set of irrational numbers in the same interval. Let X be a random variable that takes the two values 0 and 1 with

$$P(X = 1) = \begin{cases} \theta & \text{if } \theta \in A, \\ 1 - \theta & \text{if } \theta \in B. \end{cases}$$

If r is the total number of 1's in a set of n random observations on X, then from the rationality of r/n it follows at once that the maximum likelihood estimator of θ is r/n. But r/n converges (in probability) to θ or $1 - \theta$ according as $\theta \in A$ or $\theta \in B$.

Now, since A and B are both countable sets, it follows [1] that there exists a consistent test for the composite hypothesis $\theta \in A$ against the composite alterna-

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tive $\theta \varepsilon B$. In other words, there exists a function φ_n of the first n observations such that φ_n takes only the two values 0 and 1 and such that

$$P(\varphi_n = 1 | \theta) \to \begin{cases} 0 & \text{if } \theta \varepsilon A, \\ 1 & \text{if } \theta \varepsilon B, \end{cases} \qquad n \to \infty.$$

It then follows readily that the estimator

$$t_n = (1 - \varphi_n) r/n + \varphi_n (1 - r/n)$$

is a consistent estimator of θ. Thus, though there exist consistent estimators for θ, the maximum likelihood estimator is not consistent.

For a simple construction for the function φ_n, let $\{\alpha_i\}$ and $\{\beta_i\}$, for $i = 1, 2, \cdots$, be two enumerations of the sets A and B, respectively, and let δ_k be the distance between the two sets $(\alpha_1, \cdots, \alpha_k)$ and $(1 - \beta_1, \cdots, 1 - \beta_k)$. Let $k(n)$ be the largest integer k such that $\delta_k > n^{-1/4}$. Note that $k(n)$ increases monotonically to infinity. Let I_{kn} and J_{kn} be the open intervals of length $n^{-1/4}$ centered around α_k and $1 - \beta_k$ respectively and let

$$R_n = \bigcup_{k \leq k(n)} I_{kn}, \qquad S_n = \bigcup_{k \leq k(n)} J_{kn}.$$

For every n, the sets R_n and S_n are clearly disjoint. Now consider a fixed k. For all n for which $k(n) \geqq k$ we have

$$\begin{aligned} P(r/n \, \varepsilon \, S_n \mid \theta = \beta_k) &\geqq P(r/n \, \varepsilon \, J_{kn} \mid \theta = \beta_k) \\ &= P(|r/n - [1 - \beta_k]| < -n^{-1/4} \mid \theta = \beta_k) \\ &\to 1 \text{ as } n \to \infty, \end{aligned}$$

because r/n is asymptotically normal with mean $1 - \beta_k$ and asymptotic s.d. $\sqrt{\beta_k(1 - \beta_k)/n}$. By the same argument we have

$$\begin{aligned} P(r/n \, \varepsilon \, S_n \mid \theta = \alpha_k) &\leqq 1 - P(r/n \, \varepsilon \, R_n \mid \theta = \alpha_k) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, if we define

$$\varphi_n = \begin{cases} 1 & \text{if } r/n \, \varepsilon \, S_n \\ 0 & \text{otherwise,} \end{cases}$$

then φ_n clearly has the required property.

Acknowledgement. I wish to thank Mr. J. L. Hodges, Jr., for his help in the writing of this note.

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A NOTE ON SIMULTANEOUS CONFIDENCE INTERVALS

By Meyer Dwass

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Scheffé [1] and Bose and Roy [2] have recently given a method for making confidence interval estimates useful in the analysis of variance. Here we show that we can append to such a set of confidence interval estimates an additional one on D, where D^2/σ^2 is the "distance function" which determines the power of the analysis of variance test. For completeness, we first determine the same set of simultaneous confidence intervals as in [1] and [2] in a simple way. This may be of at least pedagogic interest.

Lemma. *Let $x_1, \cdots, x_k$ stand for a point in k-dimensional Euclidean space. The equations of the two planes normal to a vector $a_1, \cdots, a_k$ and tangent to a sphere of radius R centered at $c_1, \cdots, c_k$ are given by*

$$\sum a_i x_i = \sum a_i c_i + R(\sum a_i^2)^{1/2}, \qquad \sum a_i x_i = \sum a_i c_i - R(\sum a_i^2)^{1/2}.$$

This is an elementary geometric fact which we will not prove.

The analysis of variance problem can be formulated in terms of

$$y = (y_1, \cdots, y_n),$$

a set of n independent, normal random variables, each with variance σ^2, and of $\hat{\pi}_1(y), \cdots, \hat{\pi}_k(y)$, which are k independent random variables, each a linear function of $y_1, \cdots, y_n$. Each $\hat{\pi}_i$, for $i = 1, \cdots, k$, is the linear estimate of minimum variance of $E\hat{\pi}_i = \pi_i$; the variance of $\hat{\pi}_i$ is σ^2. Let ms^2 be independent of

$$\hat{\pi}_1, \cdots, \hat{\pi}_k$$

and distributed as the sum of squares of m independent $N(0, \sigma)$ random variables. The problem is to estimate or test hypotheses about $\pi_1, \cdots, \pi_k$ or linear combinations of them.

Let F_α be a number such that $(0, F_\alpha)$ includes $1 - \alpha$ of the probability of the F distribution with (k, m) d.f. Let C be a k-sphere centered at $\hat{\pi}_1, \cdots, \hat{\pi}_k$ with radius $(kF_\alpha s^2)^{1/2}$. For a vector $a = a_1, \cdots, a_k$, let $P_1(a)$ and $P_2(a)$ be the two planes which are normal to a and tangent to C. Then

$$(1) \qquad 1 - \alpha = \Pr\left\{\sum_{i=1}^{k} \frac{(\hat{\pi}_i - \pi_i)^2}{ks^2} \leqq F_\alpha\right\},$$

which is the probability that C covers the point $\pi_1, \cdots, \pi_k$. This in turn is the probability that $\pi_1, \cdots, \pi_k$ is located between $P_1(a)$ and $P_2(a)$ for all vectors a. Thus

$$(2) \quad 1 - \alpha = \Pr\{\sum a_i\hat{\pi}_i - (kF_\alpha \sum a_i^2 s^2)^{1/2} \leqq \sum a_i\pi_i \leqq \sum a_i\hat{\pi}_i + (kF_\alpha \sum a_i^2 s^2)^{1/2} \text{ for all vectors } a\}.$$

The last step follows from the lemma.

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We can characterize (2) as saying: Let $\hat{\pi}$ be the linear estimate of minimum variance of π, which is a linear combination of $\pi_1, \cdots, \pi_k$, and let $\hat{v}(\pi)$ be the estimate of the variance of $\hat{\pi}$ based on s^2. Then the confidence interval statements

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2},$$

simultaneously for all π, are correct with probability $1 - \alpha$. This result is contained in [1] and [2].

The quantity $D^2 = \sum \pi_i^2$ is a conventional measure of the "distance" of the null hypothesis that $\pi_1 = \cdots = \pi_k = 0$ from the true state of nature. The power of the analysis of variance test depends only on D^2/σ^2. Hence it would be useful for the experimenter to obtain some information about D.

Making use of the triangle inequality, it follows from (1) that

$$1 - \alpha \leqq \Pr\{(\sum \hat{\pi}_i^2)^{1/2} - (kF_\alpha s^2)^{1/2} \leqq D \leqq (\sum \hat{\pi}_i^2)^{1/2} + (kF_\alpha s^2)^{1/2}\}.$$

The quantity $\sum \hat{\pi}_i^2 = Q_1^2$ is what the experimenter calculates as the "sum of squares due to hypothesis." Hence, instead of just making the statements about the functions π, we can make the simultaneous estimates

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2}, \qquad \text{for all } \pi = \sum a_i \pi_i,$$

$$Q_1 - (kF_\alpha s^2)^{1/2} \leqq D \leqq Q_1 + (kF_\alpha s^2)^{1/2},$$

with the probability of being correct equal to $1 - \alpha$.

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AN INEQUALITY ON POISSON PROBABILITIES

By Henry Teicher

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This note proves an inequality concerning the exponential series or Poisson distribution, however one prefers to view the matter. Specifically, it will be shown that if $[\lambda]$ is the greatest integer not exceeding λ,

1) $$\sum_{j=0}^{[\lambda]} \frac{\lambda^j}{j!} > \begin{cases} e^{\lambda-1} & \text{for all } \lambda \geqq 0; \\ \frac{1}{2}e^{\lambda} & \text{for all integral } \lambda > 0. \end{cases}$$

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There are numerous ways of expressing this inequality. It will be demonstrated in the form

$$2) \qquad A_\lambda = \Pr\{X_\lambda \leqq \lambda\} > \begin{cases} e^{-1} & \text{for all } \lambda \geqq 0; \\ \frac{1}{2} & \text{for all integral } \lambda > 0. \end{cases}$$

where X_λ denotes a Poisson chance variable with parameter $\lambda \geqq 0$. The function A_λ is discontinuous at the positive integers but enjoys certain monotonic properties which may be exploited by defining

$$3) \qquad A_{n,\lambda} = \sum_{j=0}^{n} \frac{\lambda^j}{j!} e^{-\lambda} \qquad \text{for all integral } n \geqq 0.$$

Then $A_{n,\lambda} = A_\lambda$ for $n \leqq \lambda < n + 1$. As

$$\frac{d}{d\lambda} A_{n,\lambda} = \frac{-\lambda^n e^{-\lambda}}{n!} < 0,$$

we have

$$4) \qquad A_{n,n} - A_{n,n+1} = \int_n^{n+1} \frac{\lambda^n e^{-\lambda}}{n!}\, d\lambda = c_n, \text{ (say).}$$

On the other hand,

$$5) \qquad A_{n+1,n+1} - A_{n,n+1} = \frac{e^{-(n+1)}(n+1)^{n+1}}{(n+1)!} = d_n, \text{ (say).}$$

Thus to prove that $A_{n,n} > A_{n+1,n+1}$, it suffices to show that $c_n > d_n$.

Now for $0 < y < 1$, the quantity $[1 - y/(n+1)]^n$ decreases monotonically, whence $[1 - y/(n+1)]^n > e^{-y}$ in $(0, 1)$. It follows that

$$6) \qquad \int_0^1 (n + 1 - y)^n e^{-(n+1-y)}\, dy > (n+1)^n e^{-(n+1)}.$$

With a change of variable $\lambda = n + 1 - y$, we conclude that $c_n > d_n$, that is $A_{n,n} > A_{n+1,n+1}$.

In an analogous manner, $[1 + x/n]^n < e^x$ for $x > 0$ and from

$$\int_0^1 (1 + x/n)^n e^{-x}\, dx < 1,$$

we have

$$7) \qquad \int_n^{n+1} \frac{\lambda^n}{n!} e^{-\lambda}\, d\lambda < \frac{e^{-n} n^n}{n!}.$$

That is, $A_{n,n} - A_{n,n+1} < A_{n,n} - A_{n-1,n}$. Thus, in summary, we have

$$8) \qquad A_{n,n} > A_{n+1,n+1}, \qquad A_{n-1,n} < A_{n,n+1}.$$

From well known limit theorems,

$$9) \qquad \lim_{\lambda\to\infty} A_\lambda = \lim_{\lambda\to\infty} \Pr\{X_\lambda \leqq \lambda\} = \lim_{\lambda\to\infty} \Pr\left\{\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \leqq 0\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-y^2/2}\, dy = \tfrac{1}{2}.$$

The second portion of inequality (2) follows directly from (8) and (9). The remainder is a consequence of these, of the fact that $A_{n,\lambda}$ decreases monotonically from $A_{n,n}$ to $A_{n,n+1}$ in the interval $[n, n+1)$, and of

$$10) \qquad A_{n,n+1} > A_{0,1} = e^{-1}, \qquad n = 1, 2, \cdots.$$

If an additional term is included in the sum, we note that

$$11) \qquad b_\lambda = \sum_{j=0}^{[\lambda]+1} \frac{\lambda^j e^{-\lambda}}{j!} \geqq \tfrac{1}{2} \qquad \text{for all } \lambda \geqq 0,$$

since $b_\lambda > A_{n+1,n+1}$ for $\lambda \leqq n+1$. A final reformulation of part of (2) is

$$12) \qquad \Pr\{X_\lambda \leqq \lambda\} > \Pr\{X_\lambda > \lambda\}, \qquad \text{all integral } \lambda > 0.$$

NOTE ON LINEAR HYPOTHESES WITH PRESCRIBED MATRIX OF NORMAL EQUATIONS

By John E. Maxfield and Robert S. Gardner

Naval Ordnance Test Station Inyokern,

The existence theorem proven in this note relates to the problem of finding an experimental design leading to the analysis determined by the given rational matrix A of the normal equations. The matrix B found by the method used in the proof always has an interpretation as specifying the rational values of some set of regression variables. In the interesting case in which the entries of A are integers, so are the entries of B, but B is not in general interpretable as an analysis of variance. The transpose of a matrix A will be denoted by A^T.

Theorem. *Let A be a symmetric positive semidefinite matrix with rational integral entries. There exist a rational integer a and a matrix B having rational integral entries such that $BB^T = a^2A$.*

Proof. There exists a nonsingular matrix P such that $P^TAP = D$, a diagonal matrix, where P and D have rational entries ([1], p. 56). Then $(P^T)^{-1}$ has rational entries. Let a_1 be the least common denominator of the entries of P^T, a_2 of $(P^T)^{-1}$, and a_3 of D. Let $a^{1/2} = a_1a_2a_3$. Then $a^{1/2}P^Ta^{1/2}AP = aD$. If A is positive semidefinite, then aD has only positive integers or zeros on its diagonal. Let B_1 be a $n \times 4n$ matrix composed of four diagonal $n \times n$ matrices placed side by side,

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$(S_1S_2S_3S_4)$, such that $ad_{ii} = \sum_{k=1}^{4} {}_k s_{ii}^2$, where ${}_k s_{ii}$ is the entry in the ith column of S_k. This can always be done by the 4 square problem ([2], p. 235). Let $B = a^{1/2}(P^T)^{-1}B_1$, then B has integral entries, since B_1 and $a^{1/2}(P^T)^{-1}$ have integral entries. Then

$$P^TB = a^{1/2}B_1, \qquad P^TBB^TP = aB_1B_1^T = a^2D,$$

Thus $BB^T = a(P^T)^{-1}a^{1/2}Da^{1/2}P^{-1}$. But $A = (P^T)^{-1}DP^{-1}$. Thus $BB^T = a^2A$.

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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Iowa City Meeting of the Institute, November 26–27, 1945)

1. **The Distribution of the Number of Components of a Random Mapping Function. (Preliminary Report.)** Leo Katz, Michigan State College, and Jay E. Folkert, Michigan State College and Hope College.

A set H, of N elements, is mapped into itself by a function f. The most general function takes each point into an arbitrary number of points of the set. A function is said to be random if the r_i images of the point x_i may with equal probability be any subset of r_i points of H. A subset h of H is a *component* of the mapping if it is a minimal subset such that $f(h) \subset h$ and $f^{-1}(h) \subset h$. Every mapping f decomposes H into a number of disjoint components. The probability distribution of the number of components of a random mapping, where only the numbers of images of each point are known in advance, is obtained. The probability distribution of the number of components is also obtained for a variant case in which the mapping is hollow in the sense that no point maps into itself. The two distributions are obtained through a modification of the King-Jordan-Frechét formula. For each case two specializations are considered; first, one in which the multiplicity of images is the same for each point of the set, and second, where this common multiplicity is unity (so that the function f is single-valued). Numerical examples and approximations to the exact distribution are considered. This work was supported by the Office of Naval Research.

2. **Approximate Sequential Tests for Hypotheses about the Proportion of a Normal Population to One Side of a Given Number.** William Kruskal, University of Chicago.

It is sometimes of interest to test the hypothesis that the proportion of a given population exceeding a given number U is p_0 against the hypothesis that this proportion is p_1. This testing situation has been called that of testing for one-sided fraction defective. If the population is normal then the problem is to test the hypothesis $(U - \mu)/\sigma = K_0$ against the hypothesis $(U - \mu)/\sigma = K_1$. (Here μ is the mean, σ^2 the variance, and K_i the unit-normal deviate exceeded with probability p_i.) A simple translation puts this in the form: $\mu/\sigma = K_0$ vs. $\mu/\sigma = K_1$. If a sequential test is desired, it is very reasonable to base it on the sequence of Student t values computed from the first n observations. Application of the Wald sequential probability-ratio method to this sequence gives a procedure that may be called the WAGR test (after Wald, Arnold, Goldberg, and Rushton). Another sequential test

procedure for this problem has been suggested by Wallis. A sequence of successive approximations to the WAGR test is derived; of these the Wallis procedure is the final approximation. When the true value of $(U - \mu)/\sigma$ is distant from K_0 and K_1 any of these sequential procedures may lead to unduly large average sample sizes. A suggestion is made to overcome this difficulty.

3. A Note on Some Limit Theorems in Probability Theory. JOHN GURLAND, Iowa State College.

A theorem by Cramér on the limiting distribution function of a sequence of random variables is extended by widening the class of functions to which the theorem applies. A theorem by Mann and Wald is modified by relaxing the continuity restrictions. The present note utilizes the continuity theorem for the limit of a sequence of characteristic functions in establishing the results stated.

4. Uniqueness of Latin Square Association Schemes for Partially Balanced Incomplete Block Designs. (Preliminary Report.) D. M. MESNER, Purdue University.

In Latin square type partially balanced incomplete block designs with i constraints, first associates are defined by means of a square array of the n^2 varieties with $i - 2$ mutually orthogonal Latin squares. $n_1 = i(n - 1)$ and $p_{11}^2 = i(i - 1)$, determining n_2 and the remaining p_{lm}^j. For $i = 1$ the designs reduce to a special case of Group Divisible, for which Bose and Connor proved a uniqueness theorem (*Ann. Math. Stat.*, Vol. 23 (1952), p. 368). The analogous theorem here, that a design with the given parameter values implies the existence of the square array, is proved for $i = 2$. In this case, an arbitrary pair of first associates have as common first associates $n - 2$ additional varieties; these n varieties are assigned to one row of the array. Investigation of the incidence matrix of first associates shows that they are pairwise first associates. This is then used to show that the remaining association relations are those defined by the remainder of the array. The cases $i \geqq 3$ are being investigated and partial results have been obtained. That a complete generalization is impossible is shown by examples for which designs exist but the Latin squares do not. One example corresponds to the non-existent 6×6 Graeco-Latin square.

5. On Bounds for the Normal Integral. J. T. CHU, University of North Carolina.

Let $v(x) = \int_0^x (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt$, and $u(x, a) = \frac{1}{2}(1 - e^{-ax^2})^{\frac{1}{2}}$, for all $x \geqq 0$. Then $u(x, a) \leqq v(x) \leqq u(x, b)$ if and only if $0 \leqq a \leqq \frac{1}{2}$, and $b \geqq 2/\pi$. Therefore $u(x, \frac{1}{2})$ and $u(x, 2/\pi)$ are respectively the best possible lower and upper bounds for $v(x)$ of the type $u(x, a)$. The ratio $v/u(\frac{1}{2})$ is a steadily decreasing function of x and has an upper bound $2/\pi^{\frac{1}{2}}$. If $w(x, a) = \frac{1}{2}[ax^2/(1 + ax^2)]^{\frac{1}{2}}$, then $v(x) \geqq w(x, a)$ if and only if $0 \leqq a \leqq 2/\pi$, and $w(x, 2/\pi)$ is the best possible one of the type $w(x, a)$. Comparisons are made of the two new lower bounds for $v(x)$ and several known ones converted from inequalities for Mills' ratio, obtained by R. D. Gordon, Z. W. Birnbaum, and R. F. Tate (*Ann. Math. Stat.* (1941), 364–366; (1942), 245–246; (1953), 132–134). Of the three: $w(x, 2/\pi)$, $u(x, \frac{1}{2})$, and Tate's $(\frac{1}{4} + (e^{-x^2}/2\pi x^2))^{\frac{1}{2}} - (e^{-x^2/2}/x(2\pi)^{\frac{1}{2}})$, it is found that approximately the first is the best for $0 \leqq x \leqq 1$, the second, for $1 \leqq x \leqq 1.01$, and the last, for $x \geqq 1.01$.

6. Rules for Determining Error Terms in Hierarchical and Partially Hierarchical Models, MARY LUM, Wright Air Development Center.

In the classical analysis of variance procedure a linear mathematical model is used, together with certain well-known assumptions, as a basis for determining proper test

factors for the F-test of significance. This paper is concerned with models for which one or more factors are of the "within" or "nesting" type, with complete replication for the other factors. A set of rules is given for obtaining the appropriate test factors (error terms) for all main effects and interactions of the general n-factor partially hierarchical model. These rules take into account models which are fixed, mixed, or random in nature. The usual assumptions of the analysis of variance are made, with one exception: a weaker set of restrictions is imposed concerning the independence of the interaction effects. The resulting test factors are compatible with those which result from the randomization theory approach of Kempthorne and Wilk. To illustrate the convenience and usability of these rules, two examples are given.

(*Abstracts of papers presented at the Annual Membership Meeting, December 27–30, 1954*)

7. **Maximum Likelihood Estimates of Monotone Parameters.** H. D. BRUNK, University of Missouri.

The maximum likelihood estimators of distribution parameters subject to certain order relations are determined for simultaneous sampling from a number of populations, when (i) the order relations may be specified by regarding the distribution parameters, of which one is associated with each population, as values at specified points of a function of n variables monotone in each variable separately; and (ii) the distributions of the populations from which sample values are taken belong to the exponential family (cf. Girshick and Savage, "Bayes and minimax estimates for quadratic loss functions," *Proc. Second Berk. Symposium*, Univ. of Calif. Press, 1951, pp. 53–73; p. 65). The results of the present paper generalize those of Ayer, Brunk, Ewing, Reid, and Silverman, "An empirical distribution function for sampling with incomplete information," to appear in *Ann. Math. Stat.*, which treats the special case in which $n = 1$ and the distribution is binomial. This paper also represents a specific application of results obtained by Brunk, Ewing, and Utz on minimizing integrals in certain classes of monotone functions (to be offered for publication). This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

8. **Extension of Certain Classes of Contagious Distributions.** JOHN GURLAND, Iowa State College.

Beall has recently shown that Neyman's types A, B, and C of contagious distributions are members of a certain general class of contagious distributions. The present paper provides alternative ways of extending this class of distributions, and discusses the interrelations which exist among various classes. The methods employed are similar to those in recent articles on the subject.

9. **Recurrent Markov Processes II. (Preliminary Report.)** T. E. HARRIS, The Rand Corporation.

All sets considered are Borel measurable. For X a real set, let $P^n(x, E) = \text{Prob}\,(x_n \in E) \mid x_0 = x)$ for $n = 0, 1, \cdots$ be the iterates of a temporally homogeneous Markov transition function; m is a sigma-finite measure. Condition 1: $m(B) > 0$ implies $\text{Prob}\,[x_n \in B \text{ i.o.} \mid x_0] = 1$, for all $x_0 \in X$. Condition 1 implies (abstract, Nov. 27, 1954 AMS meeting, Los Angeles) existence of a sigma-finite measure π where $\pi(X) = \infty$ is allowed, such that m is absolutely continuous with respect to π, with $\pi(E) = \int_X \pi(dx)P^1(x, E)$, for all $E \subset X$. Next, Condition 1 being satisfied, suppose $A \subset X$, with $0 < \pi(A) < \infty$; suppose the Markov process, obtained by observing only those x_n which are in A, satisfies condition

D_0 (Doob, *Stochastic Processes*, p. 221). Then

$$\lim_{N\to\infty} \Sigma_{n=1}^{N} P^n(x, E)/\Sigma_{n=1}^{N} P^n(y, F) = \pi(E)/\pi(F)$$

for all x, y, E, F in A, provided $\pi(F) > 0$.

10. **The Role of Perks' Distributions in the Theory of Wiener's Stochastic Function.** JOSEPH TALACKO, Marquette University and University of California.

In "Wiener's Random Function, and other Laplacian Random Functions," by Paul Lévy, *Second Berkeley Symposium on Math. Stat. and Prob.*, 1951, pp. 171–187, Wiener's function has been applied to the study of the Brownian plane curve. For the area between an arc and its chord particular solutions of the relative transition probability are found. This paper, using the method of characteristic functions, shows that the solutions are all non-Gaussian and from the Perks family of functions $f(x) = c(e^{\pi x} + k + e^{-\pi x})^{-1}$, where $0 \leqq k \leqq 2$.

11. **On the Orthogonality of Measures and the Existence of Consistent Tests.** CHARLES KRAFT, University of California.

Let $(\mathfrak{X}, \mathfrak{A}) = \Pi_{i=1}^{\infty} (\mathfrak{X}_i, \mathfrak{A}_i)$ and let $\{P\}$ and $\{Q\}$ be any two families of measures on $\mathfrak{A}$. $\{P\}$ is called orthogonal to $\{Q\}$ when there is a set A in $\mathfrak{A}$ such that $P(A) = 1$ and $Q(A) = 0$ for all $P \in \{P\}$ and $Q \in \{Q\}$. When the families each contain one element, their orthogonality is equivalent to the existence of a consistent test between the simple hypotheses P and Q. Further, the orthogonality can be characterized in terms of "distances" between the n-dimensional distributions. This characterization is a generalization of the one given by Kakutani for product measures. It is shown that the orthogonality of the two families of measures and the existence of a consistent test for the corresponding hypotheses are identical as long as the families are countable, but that this is not necessarily the case when the families are not countable. The methods used for these problems are then applied to the problem of consistency for likelihood ratio test and maximum likelihood estimates for cases when the observations are dependent.

12. **The Minimax Character of the Neyman-Pearson Critical Region.** L. M. COURT, Diamond Ordnance Fuze Laboratories.

In his "Statistical Decision Functions", Wald indicates that the Neyman-Pearson theory of testing statistical hypotheses is a special case of Decision Theory for which $d(0) \in D^* = (1, \cdots, N)$ and $d(x \mid s_1) \in D^1 = (d_1^1, d_2^1)$. By a further natural restriction of the class of decision functions to be considered, the minimax character of the Neyman-Pearson critical region is easily established. Since a minimax solution of a decision problem is essentially also a Bayes' solution with respect to a least favorable a priori distribution, the Neyman-Pearson critical region also corresponds to a Bayes' solution.

13. **On the Existence of Linear Regressions.** T. S. FERGUSON, University of California.

Let ξ and η be independent nondegenerate random variables with zero means. If the regression of ξ on $X = a\xi + \eta$ is linear for two values, a_1 and a_2, of a with $|a_1| \neq |a_2|$, then ξ and η are both semistable in the sense of P. Lévy, with the same characteristic exponent. Hence if second moments are assumed, both ξ and η are normal. If the regression of ξ on X is linear for three values a_1, a_2, and a_3 of a, with $\log |a_1/a_2|$ and $\log |a_1/a_3|$ incommen-

surable, then both ξ and η are stable. This extends the results of E. Fix: "Distributions which lead to linear regressions," *Proc. Berkeley Symposium on Math. Stat. and Prob.*, Univ. of Calif. Press, 1949. Two further results are obtained. (1) If the regression of ξ on $X_1 = a_1\xi + \eta_1$ and $X_2 = a_2\xi + \eta_2$ is linear where ξ, η_1, and η_2 are independent nondegenerate random variables with zero means, and $a_1 \neq 0 \neq a_2$, then ξ, η_1, and η_2 are each normal. (2) Let $X = A\xi + \eta$ where ξ and η are independent s- and n-dimensional random vectors, respectively, with zero means, and A is an $n \times s$ matrix of constants. The components of η are assumed independent and at least one of them nondegenerate. If the regression of $b\xi$ on X is linear for all values of the matrix A and the vector b, then for $n = 1$ both ξ and η are multivariate stable and for $n > 1$ both ξ and η are multivariate normal.

14. On the Asymptotic Distribution of U-Statistics Modified by the Introduction of Estimates of Parameters. (Preliminary Report.) B. V. Sukhatme, University of California.

Let $X_1, \cdots, X_n$ be n independent identically distributed r.v.'s with c.d.f. $F(x - \theta)$. Let $\hat{\theta}_n(X_1, \cdots, X_n)$ be an estimate of θ such that given $\epsilon > 0$, there exists a number b such that for n sufficiently large, $P\{|\hat{\theta}_n - \theta| \geqq b/\sqrt{n}\} \leqq \epsilon$. The paper investigates the asymptotic behavior of U-statistics when each observation is centered at $\hat{\theta}_n$ and shows that under certain mild restrictions, they are asymptotically normally distributed. Conditions are also given under which these statistics have the same asymptotic normal distribution as the original U-statistics. These results have been extended to random vectors, to functions of several U-statistics, and to the type of statistics that occur in testing the hypothesis that two populations differ only in location.

15. Essentially Complete Classes of Experiments. Sylvain Ehrenfeld, Columbia University.

An experimenter has available a family of independent random variables Y_x depending on a parameter $\theta \in \Omega \subseteq E^{(p)}$ and where $x \in A \subseteq E^{(k)}$, with A compact and $E^{(p)}$ and $E^{(k)}$ Euclidian spaces. This paper deals with choosing the x's if the experimenter is restricted to a finite number of observations. Suppose Y_x is normally distributed with mean $\psi(\theta, x)$ and variance σ^2. Let $\mathcal{E}_N(B)$ be the class of experiments with x restricted to $B \subseteq A$, and let $V_{x_1,\cdots,x_N}(\widehat{t'\theta})$ be the variance of the maximum likelihood estimate of $t'\theta$ when $Y_{x_1},\cdots,Y_{x_N}$ are observed. $\mathcal{E}_N(B)$ is said to be essentially complete if for any x_j $(j = 1, \cdots, N)$ with $x_j \in A$ and any unknown $\theta \in \Omega \subseteq E^{(p)}$ there exists a set x_j^* $(j = 1, \cdots, s)$ with $s \leqq N$ and $x_j^* \in B \subseteq A$ such that $V_{x_1^*,\cdots,x_s^*}(\widehat{t'\theta}) \leqq V_{x_1,\cdots,x_N}(\widehat{t'\theta})$ for *all* $t \in E^{(p)}$. The notion of asymptotic essential completeness is defined and several classes are shown to have the above property for this general case. In case $\psi(\theta, x) = x'\theta$ (the linear Hypothesis) stronger results can be proven without the normality assumption. Let $r(A) \subseteq A$ be defined as follows: $r(A) = \{\lambda(x)x \mid x \in A\}$ where $\lambda(x)$ is for each $x \in A$ $(x \neq 0)$ a scalar such that $\lambda(x) \geqq 1$ and $\lambda(x)x \in A$, while if $\lambda' > \lambda(x)$ then λ' non-$\in A$. It is shown that: (a) $\mathcal{E}_N(r(A))$ is essentially complete for all N; (b) if $b_1, \cdots, b_s$ are s vectors in $r(A)$ such that their convex closure contains $r(A)$, then $\mathcal{E}_N(b_1, \cdots, b_s)$ is asymptotically essentially complete; (c) with $b_1, \cdots, b_s$ as above $\mathcal{E}_{N+s}(b_1, \cdots, b_s)$ is essentially complete with respect to $\mathcal{E}_N(A)$ for all N.

16. A Theorem on Uniform Convergence of Families of Sequences of Random Variables. Emanuel Parzen, Columbia University.

In the author's paper "On uniform convergence of families of sequences of random variables" (*Univ. Calif. Publ. Statist.*, Vol. 2 (1954), pp. 23–54), a theorem (18D) was stated which is a uniform analogue of theorem 20.6 in Cramér's *Mathematical Methods of Statistics*

(Princeton, 1946). In this note we state a theorem, more general than our previous theorem, which pays attention to UC^* convergence, and does not impose unnecessarily broad equicontinuity conditions on the limit distribution functions. This theorem applies to the multivariate case as well as to the univariate case, and may be proved as an immediate consequence of the Uniform Continuity Theorem. ASSUMPTIONS: Let r-dimensional random vectors $\mathbf{X}^{(n)}(\theta) \xrightarrow{UC^*} \mathbf{X}(\theta)$, and s-dimensional random vectors $\mathbf{Y}^{(n)}(\theta) \xrightarrow{P} \mathbf{a}(\theta)$, constants, uniformly in θ. For $j = 1, \cdots, m$, let $u_j(\mathbf{x}, \mathbf{y}, \theta)$ be a family of real-valued Borel functions, such that, for each $\mathbf{x}$, $u_j(\mathbf{x}', \mathbf{y}, \theta) \to u_j(\mathbf{x}, \mathbf{a}(\theta), \theta)$ uniformly in θ as $\|\mathbf{x}' - \mathbf{x}\| \to 0$ and $\|\mathbf{y} - \mathbf{a}(\theta)\| \to 0$, and, for each $\mathbf{x}$, $|u_j(\mathbf{x}, \mathbf{a}(\theta), \theta)|$ is bounded uniformly in θ. Define $\mathbf{u}(\mathbf{x}, \mathbf{y}, \theta) = (u_1(\mathbf{x}, \mathbf{y}, \theta), \cdots, u_m(\mathbf{x}, \mathbf{y}, \theta))$. Let $\mathbf{Z}^{(n)}(\theta) = \mathbf{u}(\mathbf{X}^{(n)}(\theta), \mathbf{Y}^{(n)}(\theta), \theta)$ and $\mathbf{Z}(\theta) = \mathbf{u}(\mathbf{X}(\theta), \mathbf{a}(\theta), \theta)$. CONCLUSION: $\mathbf{Z}^{(n)}(\theta) \xrightarrow{UC^*} \mathbf{Z}(\theta)$.

17. **Decision Procedures for the Comparison of Exponential Populations.** F. S. McFEELY and J. E. FREUND, Virginia Polytechnic Institute.

Minimax procedures are applied to the comparison of two exponential populations to determine optimum sample size if one of two alternatives must be accepted. Interpreting the first r out of n ordered observations from each population as failures in a life test, the loss function is taken as

$$L = c_1(1 - \theta_1/\theta_2) \Pr\{\hat{\theta}_1 > \hat{\theta}_2\} + 2c_3 r + 2c_2'(n - r) + c_3\theta_2 \textstyle\sum_{j=1}^{r}(n - j + 1)^{-1}$$

assuming that $\theta_2 > \theta_1$. In order, the terms stand for losses due to wrong decision, to items used, and to cost of experimentation. Boundedness is obtained by considering L to be split into components L_1 and L_2 consisting of the first two and last two terms of L respectively. Minimax procedures applied to L_1 yield an r which is subsequently used in minimizing L_2 for n. Values of r and n are tabulated for selected values of the constants in L.

18. **The Efficiency of Some Nonparametric Competitors of the t-test.** J. L. HODGES, JR. and E. L. LEHMANN, University of California.

Consider samples from continuous distributions $F(x)$ and $F(x - \theta)$. We may test the hypothesis $\theta = 0$ by using the two-sample Wilcoxon test. It is shown that its asymptotic efficiency (in the sense of Pitman), relative to the t-test, never falls below 0.864. This lower bound is attained when $F'(x)$ is parabolic. A number of alternative notions of asymptotic efficiency are compared. In particular, the limit of the power efficiency function of Dixon at a fixed alternative is derived for the sign test relative to the t, for normal F.

19. **Multi-level Continuous Sampling Plans.** GERALD LIEBERMAN, Stanford University, and HERBERT SOLOMON, Teachers' College, Columbia University.

In 1943, Dodge (*Ann. Math. Stat.*, Vol. 14 (1943), pp. 264–279) published a sampling inspection plan for continuous production for production in statistical control. Wald and Wolfowitz (*Ann. Math. Stat.*, Vol. 16 (1945), pp. 30–49) later gave a continuous sampling plan which guaranteed a prescribed Average Outgoing Quality Limit (AOQL) even when production is not in statistical control. An inconvenient feature of both plans is the abrupt change between 100 percent inspection and partial inspection. Multi-level continuous sampling plans are introduced to provide smoother transitions between partial inspection and 100 percent inspection. The Dodge plan is a special case occurring when there is one sampling level. The Average Outgoing Quality (AOQ) function is derived for a k-level plan. Contours of constant AOQL are developed for two-level sampling plans and infinite level

sampling plans. An approximation is given for contours of constant AOQL for other levels. The choice of a specific multi-level plan is discussed in terms of minimum average fraction inspected and local stability.

20. Numerical Values for Covariances of Order Statistics for Samples of Size Twenty and Less from the Normal Distribution. D. TEICHROEW, University of California.

This paper gives the covariances to more than ten decimal places. The method of computation is given and auxiliary tables obtained in the process are described. These tables include the normal density function, the normal probability integral and its first nineteen powers. All these functions are now available to twenty-five decimal places for the argument increasing in steps of .02.

The table of covariances is the end result of a project proposed by W. J. Dixon to the Institute for Numerical Analysis of the National Bureau of Standards. The project was started before and completed after the Institute for Numerical Analysis had become Numerical Analysis Research of the University of California, Los Angeles. The project was supported by the Office of Naval Research.

21. Estimation of the Parameters of the Rectangular and the Exponential Populations from Singly and Doubly Censored Samples. A. E. SARHAN, University of North Carolina.

The best linear estimates of the parameters of the rectangular population $f(y) = 1/\theta_2$, $\theta_1 - \frac{1}{2}\theta_2 \leqq y \leqq \theta_1 + \frac{1}{2}\theta_2$, and of the exponential population $f(y) = \sigma^{-1} \exp[-(y - \mu)/\sigma]$, where $\mu \leqq y \leqq \infty$, and the variances and relative efficiencies of these estimates are worked out for singly and doubly censored samples of size n. The corresponding quantities are also derived for the special cases $f(y) = 1/\theta_2$, where $0 \leqq y \leqq \theta_2$, and $f(y) = 0/\sigma^{-1}e - y/\sigma$, where $0 \leqq y \leqq \infty$.

22. Estimation from "Censored" Samples of Extreme Data. JULIUS LIEBLEIN, National Bureau of Standards.

In a previous report (NACA Technical Note 3053), the author has obtained unbiased minimum-variance order statistics estimators of an arbitrary linear function of the two parameters of the extreme-value distribution with c.d.f. $P(x) = \exp\{-\exp[-(x - u)/\beta]\}$. These estimators are "optimum" within the class of all linear functions of the order statistics of a sample of given size n. The present paper extends these methods, together with the necessary tables, to the case of a "censored" sample, defined as one where the total number of observations is known but full information is not available with regard to some of them. For example, in fatigue testing, the test may be discontinued before all test specimens have failed, so that the endurance lives for the "run-outs" are not available. The method is applied to an example of fatigue life-testing of ball bearings.

23. On the Probability Integral Associated with a Certain Multivariate Test. K. V. RAMACHANDRAN, University of North Carolina.

The acceptance region $c_0 \leqq$ all $c(S_1S_2^{-1}) \leqq c_0'$ has been proposed by Roy for testing the hypothesis H_0: $\Sigma_1 = \Sigma_2$, where Σ_1 and Σ_2 stand for the dispersion matrices of two p-variate normal populations, S_1 and S_2 for those of two random samples from them, $c(S_1S_2^{-1})$ for the characteristic roots of $S_1S_2^{-1}$, and c_0 and c_0' are two preassigned positive constants. In this paper a technique for obtaining the probability $P[c_0 \leqq \text{all } c(S_1S_2^{-1}) \leqq c_0'|H_0]$ is devel-

oped which is an extension of the previous methods due to Roy, Nanda and Pillai for obtaining $P[\text{all } c(S_1S_2^{-1}) \leqq c_0' \mid H_0]$ or $P[c_0 \leqq \text{all } c(S_1S_2^{-1}) \mid H_0]$. The actual expressions for the probability are given for $p = 2, 3, 4$ and 5.

24. Some Uses of Quasi-ranges. J. T. Chu, University of North Carolina.

Confidence intervals for, and tests of hypotheses about, the difference of two quantiles (of the same distribution) are obtained, using the difference (called quasi-range) of two properly chosen order statistics. Let $0 < p < q < 1$ be given and ξ_p and ξ_q be the pth and qth quantiles of a certain distribution. Let x_r and x_s, with $r \leqq s$, be the rth and sth order statistics in a sample of size n. Then

$$P\{(x_s - x_r) \geqq (\xi_q - \xi_p)\} \geqq B_n(s-1, q) - B_n(r-1, p) = L,$$

where $B_n(r, p) = \Sigma_{i=0}^{r}\binom{n}{i}p^i(1-p)^{n-i}$. Corresponding to every integer k, with $0 \leqq k \leqq n-1$, define $r = [(n-k)p / (1-c)] + 1$, and $s = r + k$, where $[x]$ is the integral part of x, and $c = q - p$. Then L is a nondecreasing function of k. For a given α (and sufficiently large n), choose the least integer k such that $L \geqq 1 - \alpha$. Then $P\{(x_s - x_r) \geqq (\xi_q - \xi_p)\} \geqq 1 - \alpha$. If the parent distribution satisfies certain continuity conditions, then $x_s - x_r$ is a consistent estimate of $\xi_q - \xi_p$ and variance $O(n^{-1})$. Confidence lower bounds and intervals for, and tests of hypotheses about, $\xi_q - \xi_p$ can be obtained in similar ways. Applications are given to the standard deviation of a normal distribution, quantiles, and the proportion of defectives in a lot.

25. Inadmissibility of the Usual Estimate for the Mean of a Multivariate Normal Distribution. Charles M. Stein, Stanford University.

If $X_1, \cdots, X_n$ are independently normally distributed with unknown means $\xi_1, \cdots, \xi_n$ and variance 1, the usual estimate of ξ_i is X_i. If the loss is the sum of squares of the errors, this estimator is admissible for $n \leqq 2$, inadmissible for $n \geqq 3$. In the latter case, a better estimator is $[1 - (n-2) / (c + \Sigma X_i^2)]X_i$, with c sufficiently large. The constant c should be chosen to increase with n a little faster than $\sqrt{n}$, and the estimate should be replaced by 0 if the correction factor is negative. As $\Sigma\xi_i^2 \to \infty$, this yields asymptotically the best possible improvement over the usual estimator among all estimators having spherical symmetry about the origin. For large $\Sigma\xi_i^2$ or large n, the mean squared error of the proposed estimator is nearly $n - (n-2)^2 / (\Sigma\xi_i^2 + n)$. For large n, it is nearly a Bayes' estimator with respect to any spherically symmetric a priori distribution. For extremely large n, it seems desirable to represent the parameter space as an orthogonal direct sum and apply the method separately in each of the spaces entering into the direct sum. In many experimental designs there is a natural way of doing this, for example by the order of the interaction. Better approximations are needed before this method can be applied freely. It seems likely that the present results can be extended to a large class of problems in which one is interested in estimating many parameters with a single overall measure of the error.

26. A Locally Optimal Test for the Independence of Two Poisson Variables. Mohamed S. Ahmed, University of California.

Let (x_i, y_i), for $i = 1, 2, \cdots, n$, be independent observations on a bivariate Poisson with respective expectations λ and μ. It is shown that no test for the independence of x and y can be both similar in λ, μ and also uniformly most powerful. However, a test of the form $\Sigma_1^n (x_i - \bar{x})(y_i - \bar{y}) \geqq c_n(\bar{x}\bar{y})^{\frac{1}{2}}$ is similar and locally most powerful. Using a theorem on the convergence to a normal of the conditional distribution of $\Sigma_1^n (x_i - \bar{x})(y_i - \bar{y})$, given $\bar{x}$ and $\bar{y}$, asymptotic formulae for the values of the c_n's corresponding to a given level are derived. Furthermore, it is shown that the asymptotic power of the test can be obtained from a normal approximation.

27. Statistical Estimation of the Endurance Limit.

E. J. GUMBEL, Columbia University.

The theoretical scheme for fatigue failures and life tests is the probability $l_S(N)$ of surviving N cycles under a stress S. Within any such scheme all specimens tested are expected to fail after a finite number of cycles. However, numerous observations show cases where n' out of n pieces tested survived 10^7 or 10^8 cycles, which is taken as infinity. A stress S_0 so small that the probability of surviving an infinity of cycles is equal to unity is called the endurance limit. It can exist provided there exist minimum lives $N_{0,S}$, that is numbers of cycles before which fracture cannot occur under the stress S. To explain the existence of an endurance limit, the probability of survival is written $l_S(N) = l_S(\infty) + 1 - l_S(\infty)\, L_S(N)$ where $L_S(N)$ stands for the probability of survival up to N cycles of the broken pieces and $l_S(\infty)$ is the probability of permanent survival. A first estimate for $l_S(\infty)$ is the quotient $(n - n') / (n + 1)$. If the probability $L_S(N)$ is interpreted as the asymptotic probability of the smallest values of a variate limited to the left by $N_{0,S}$, the value $l_S(\infty)$ can be estimated by the method of moments and successive approximations. Finally, the probability $l_S(\infty)$ is again interpreted by the same asymptotic theory of smallest values with the lower limit S_0, which may be estimated by a procedure similar to the estimation of the minimum lives $N_{0,S}$. The few reliable observations on the frequencies of permanent survival which are available lead to a good fit of the theory. (The work was done in part under the support of the Office of Ordnance Research.)

28. On the Efficiency of Estimates in Successive Multiphase Sampling. (Preliminary Report.)

B. D. TIKKIWAL, University of North Carolina.

The estimates given in a preliminary report, (*Ann. Math. Stat.*, Vol. 25 (1954), pp. 174), on sampling on successive occasions for K characters from a finite population of size N, are further seen to satisfy the necessary and sufficient conditions for the efficiency of an estimate given by Patterson (1950), in view of the following lemma. Let ${}_lX_{ij}$ represent the ith unit observed for the jth character on the lth occasion and ${}_l\hat{X}_j$ the best estimate, with corresponding variances ${}_l\sigma_j^2$ and $V_N({}_l\hat{X}_j)$. The correlation between the units for the same character on lth and mth occasions is $P_{l,m}$, and the correlation between units for ith and jth characters on the same occasion is P'_{ij}. Then

$$\mathrm{Cov}({}_lX_{ij}\,{}_m\hat{X}_{j'}) = \begin{cases} P'_{j,j'}({}_l\sigma_j/{}_m\sigma_{j'})\ \mathrm{Cov}({}_lX_{ij'}\,{}_m\hat{X}_{j'}), & j > j'; \\ P'_{j,j'}({}_m\sigma_{j'}/{}_l\sigma_j)\ \mathrm{Cov}({}_lX_{ij}\,{}_m\hat{X}_j), & j < j'; \\ \psi\ P_{l,m}\ {}_l\sigma_j\ {}_m\sigma_j, & j = j'. \end{cases}$$

When $l < m$, $\psi = \Pi({}_l\phi_j)[V_\infty({}_lX_j) / {}_l\sigma_j^2] + \Sigma_{i=1}^{l-1}\,[P'^2_{ij}\ V_\infty({}_l\hat{X}_i) / {}_l\sigma_i^2]\,[\Pi({}_l\phi_i) - \Pi({}_l\phi_{i+1})] - 1/N$, where limits on Π are $t = l + 1$ to $t = m$ and ${}_l\phi_j$ is the weight associated with the estimate for jth character on tth occasion. When $l \geq m$, $\psi = V_N({}_m\hat{X}_j) / {}_m\sigma_j^2$. If ${}_lX_{ij}$ is not in the sample, $\psi = -1/N$. The estimates in an infinite population are seen to satisfy the conditions of Patterson by putting $N = \infty$.

29. Identification of a Certain Stochastic Structure.

HENRY TEICHER, Purdue University.

The structure characterized by $x_i = u_i + \alpha u_{i-1}$, with the u_i identically and independently distributed, is analyzed with respect to the identifiability of the parameter α and the distribution of u. Under simple conditions both are identifiable.

30. Detection and Hypothesis Testing by Linear Methods. EMANUEL PARZEN, Columbia University.

The "measurement signal-to-noise ratio" considered by various authors, and considered as a detection criterion by E. Parzen and N. Shiren (*An analysis of a general system for the detection of amplitude modulated noise*, to be published) is shown to be a useful generalization of the notion of power signal-to-noise ratio, and to be unambiguously defined in cases where the latter is not. It is also shown that the detection criterion provides a possible risk function on which to base a theory of statistical hypothesis testing restricted to rejection regions determined by linear functionals in the manner of estimation by linear methods. The test so obtained for the mean value function of a random time function coincides with the test obtained for a Gaussian process by the Neyman-Pearson theory.

31. Probability of Indecomposability of a Set Under Random Mapping. LEO KATZ, Michigan State College.

A set Ω of N (finite) points is mapped *into* itself by a single-valued function, $f(x)$. The mapping, in the general case, is random if the N^N sample points of $f(x)$ have uniform probability. Each function decomposes the set into a number, k, of disjoint, non-null, *minimal* invariant subsets, as $\Omega = w_1 + w_2 + \cdots + w_k$, with $f(w_i) \subset w_i$ and $f^{-1}(w_i) \subset w_i$. If $k = 1$, the set is indecomposable. In the "hollow" case, of some importance in social psychology, no point maps into itself and the $(N-1)^N$ sample points have uniform probability. For both cases, the probability that the set is indecomposable under the random mapping is obtained. A table of the probabilities is given for $N = 2(1)20(2)40(5)100$. Asymptotic expressions for the probabilities indicate that indecomposability is more likely in the hollow case by a factor of e, approximately. This last finding is significant in that it contradicts standard sociometric folklore, which holds that the hollow case may be approximated by the general one, for large N. (This work was supported by the Office of Naval Research.)

32. On the Near Monotonic Character of the Power Function of a Certain Multivariate Test. S. N. ROY and K. V. RAMACHANDRAN, University of North Carolina.

In previous papers by one of the authors the acceptance region $c_0 \leqq c_1 \leqq c_p \leqq c_0'$ is proposed for the hypothesis $H_0: \Sigma_1 = \Sigma_2$, and some of its operating characteristics are studied, where Σ_1 and Σ_2 stand for the dispersion matrices of two p-variate normal populations, S_1 and S_2 for those of two random samples from them, c_1 and c_p for the smallest and largest characteristic roots of $S_1S_2^{-1}$, and $\gamma_1 \leqq \cdots \leqq \gamma_p$ for the characteristic roots of $\Sigma_1\Sigma_2^{-1}$. The quantities c_0 and c_0' are supposed to be given by (i) $P(c_0 \leqq c_1 \leqq c_p \leqq c_0' \mid H_0) = 1 - \alpha = P_0$ (say); and

$$\text{(ii)} \qquad \left[\frac{\partial P}{\partial \gamma_i}(c_0 \leqq c_1 \leqq c_p \leqq c_0' \mid H)\right]_{\gamma_1=\cdots=\gamma_p=1} = 0, \qquad i = 1, \cdots, p.$$

It is to be noted that (a) $H_0 \Leftrightarrow \gamma_1 = \cdots = \gamma_p = 1$ and $H \neq H_0$; (b) $P(c_0 \leqq c_1 \leqq c_p \leqq c_0' \mid H) = P(c_0 \leqq c_1 \leqq c_p \leqq c_0' \mid \gamma_1, \cdots, \gamma_p \neq 1) = P$ (say); and also that (c) the equations (ii) are equivalent to just one equation. In this paper it is proved (1) that if all the γ's are equal and stay equal, in other words, if $\Sigma_1\Sigma_2^{-1} = \gamma I(p)$, or $\Sigma_1 = \gamma\Sigma_2$, then P or the second kind of error monotonically decreases, that is, the power monotonically increases as γ tends away from 1 (increasing or decreasing) and also (2) that when the γ_i's are not equal the power increases as each γ_i, separately, tends away from 1, provided that (i) all are greater than 1 or less than 1 to start with and (ii) the gap between the smallest and the largest one does not exceed a certain value.

33. Simultaneous Distribution of the Mean, Standard Deviation, and Range for Probability Density Functions of Doubly Infinite Range. (Preliminary Report.) MELVIN D. SPRINGER, U. S. Naval Ordnance.

The simultaneous distribution of the sample mean ($\bar{x}$), standard deviation (S), and range (R) is derived in integral form for probability density functions $f(x)$, for $-\infty < x < \infty$. This derivation is effected through the use of a transformation which transforms the sample probability element $\prod_1^n f(x_i)dx_i$ into an element which is a function of $x_2, x_3, \cdots, x_{n-2}, \bar{x}, S, R$. After limits of integration of x_{n-r} are determined, this function is integrated with respect to x_{n-r} for $r = 2, 3, \cdots, n-2$, to obtain the simultaneous distribution $F(\bar{x}, S, R)$. Integration of $F(\bar{x}, S, R)$ with respect to $\bar{x}$, S, and R, respectively, yields the joint distributions $G(S, R)$, $H(\bar{x}, R)$, and $K(\bar{x}, S)$, whose correlation properties are then considered. In particular, the regression functions, scedastic functions, and correlation ratios are presented in general (integral) form for both regression systems of each of the joint functions $G(S, R)$, $H(\bar{x}, R)$, and $K(\bar{x}, S)$. The methods are then applied to the normal density function to illustrate the procedures for determining $F(\bar{x}, S, R)$ and for analyzing the regression system of R on S.

34. A Class of Asymptotic Tests of Composite Hypothesis. J. NEYMAN, University of California.

Let H be a composite hypothesis asserting that all the observable r.v.'s of the sequence $\{X_i\}$ (possibly multivariate) are completely independent and have the same probability density $p(x|\theta)$ defined for $x \in \mathfrak{X}$ and depending on a parameter point $\theta = \{\theta_1, \theta_2, \cdots, \theta_s\}$. H fails to specify the value θ^0 of θ except that it must belong to an open set Θ. The problem considered is that of "asymptotically α-similar" regions for testing H. Let $Y_n = \{X_1, X_2, \cdots, X_n\}$ and w_n be a subset of $\mathfrak{X}^n$. We say that the sequence $\{w_n\}$ is an asymptotically α-similar critical region for testing H if $P\{Y_n \in w_n \mid \theta^0\} \to \alpha$ as $n \to \infty$ for all $\theta^0 \in \Theta$. Under suitable regularity conditions on $p(x|\theta)$ it is proved that, whenever for each $j = 1, 2, \cdots, s$ there is known a consistent estimate $\theta_i^*(Y_n)$ of θ_i^0 converging to θ_i^0 so fast that the product $\sqrt{n}\,|\theta_i^*(Y_n) - \theta_i^0|$ is bounded in probability, a family Φ of asymptotically α-similar regions can be constructed as follows. Let $\nu(\alpha)$ be an open set of numbers such that $N(0, 1) \in \nu(\alpha)$ with probability α. Let $f(x, \theta)$ be a sufficiently regular function on $\mathfrak{X} \times \Theta$, such that the expectation of $f(X, \theta^0)$ is zero and its variance unity for all $\theta^0 \in \Theta$. Let $\varphi_j(x, \theta) = \partial(\log p)/\partial\theta_j$, let $A_j(\theta^0)$ be the regression coefficient of $f(X, \theta^0)$ on $\varphi_j(X, \theta^0)$ and let $\sigma^2(\theta^0)$ be the variance of $f(X, \theta^0) - \Sigma_1^s A_j(\theta^0)\varphi_j(X, \theta^0)$. Finally, let w_n stand for the subset of $\mathfrak{X}^n$ defined by

$$\sum_{i=1}^{n} \{f[x_i, \theta^*(Y_n)] - \sum_{j=1}^{s} A_j[\theta^*(Y_n)]\varphi_j[x_i, \theta^*(Y_n)]\} \frac{1}{\sigma[\theta^*(Y_n)]\sqrt{n}} \in \nu(\alpha).$$

Then the sequence $\{w_n\}$ is asymptotically α-similar. The problem of selecting within Φ an "optimum" critical region for testing H is thus reduced to that of selecting $f(x, \theta)$ and $\nu(\alpha)$.

35. The Transitions of a Markovian Process from Set to Set. BAYARD RANKIN, Massachusetts Institute of Technology.

Let $(\Omega, \mathfrak{A}, P)$ be a probability space, $(\Gamma, \mathfrak{B})$ a measurable space and R^+ the space of positive real numbers. A measurable function $G_\omega(t)$ defined on $\Omega \times R^+$ to Γ will be called a stochastic process with range space Γ. If γ represents a measurable partition of Γ, it is possible to describe the transitions of a Markovian process $G_\omega(t)$ from set to set of γ in terms of a stochastic process $F_\omega(t)$ with range space γ. The process $F_\omega(t)$ is such that for any

$g \in \gamma$, $F_\omega(t) = g$ whenever $G_\omega(t) \in g$. In general $F_\omega(t)$ will be non-Markovian, but analytically it may be treated as a temporally inhomogeneous Markov process. In particular, if the transitions of $G_\omega(t)$ through the points of Γ can be described by a temporally homogeneous linear diffusion equation, the transitions of $F_\omega(t)$ through the points of γ can be described by a temporally inhomogeneous linear equation which differs from the latter only in the coefficients. The new coefficients will be weighted averages of the old, the weighting functions being the conditional distribution of $G_\omega(t)$ given that $G_\omega(t)$ belongs to some specified set of γ and given the initial conditions. Thus the "past" of the non-Markovian process is put into the coefficients. If the conditional distribution is arbitrarily assumed independent of time, this defines a new stochastic process $F_\omega(t)$ which is non-Markovian but both temporally homogeneous and linear. (This work was sponsored in part by the Office of Naval Research.)

36. Confidence Intervals for the Ratio and for the Difference of Two Probabilities. (Preliminary Report.)

R. R. Bahadur and W. H. Kruskal, University of Chicago.

Let X_i for $i = 1, 2$ be the numbers of successes in binomial samples of sizes n_i with success probabilities θ_i. The authors propose specific exact randomized tests of composite null hypotheses such as $\theta_1/\theta_2 = 2$ and $\theta_1 - \theta_2 = .3$, more precisely of any linear null hypothesis of form H: $a\theta_1 + b\theta_2 + c = 0$, where a, b, and c are constants. In the usual manner, such tests may be used to generate confidence sets for parameter functions like θ_1/θ_2 and $\theta_1 - \theta_2$. The proposed tests are based on the observation that if Z is 1 or 0 with probability θ and $1 - \theta$, and if A and B have independent distributions on the same two points with probabilities for 1 of α and β respectively, then $AZ + B(1 - Z)$ also is 1 or 0 with probability for 1 equal to $\alpha\theta + \beta(1 - \theta)$. In this way null hypotheses of form H may be reached to form $\theta_1 = \theta_2$. The authors propose to study further the small sample and asymptotic properties of these tests and confidence sets. One of the authors (WHK) is studying similar methods in other areas, notably in components of variance problems.

37. Limit Theorems for Conditional Distributions.

G. P. Steck, University of California.

Let $U_n = (U_{n1}, \cdots, U_{nL})$ and $V_n = (V_{n1}, \cdots, V_{nM})$ be random vectors. Given that the limiting distribution of (U_n, V_n) exists, conditions are given which insure that the limiting conditional distribution of U_n given V_n also exists. In this connection we have the following theorems: Theorem 1. *Let the conditional characteristic function of U_n given $V_n = v$ (v a possible value of V_n) be denoted by $w_n(v)$. If* (i) *(U_n, V_n) converges in law to the random vector (U, V), and* (ii) *the family $\{w_n(v)\}$ is equicontinuous on bounded sets, then $\{w_n(v)\}$ can be extended to a family $\{w_n^*(v)\}$ such that $w_n^*(v) \to w(v)$ uniformly for v in a bounded set, where $w(v)$ is the conditional characteristic function of U given $V = v$.* Theorem 2. *Let U_n and V_n be as indicated above. If* (i) *(U_n, V_n) has a limiting normal distribution, and* (ii) *V_{nk} is the normalized sum of independent random variables which are identically distributed on a lattice independently of n, then the limiting conditional distribution of U_n given V_n is also normal.* Theorem 2 can be extended to cases where the summands of V_{nk} are not identically distributed and have distributions depending on n. An application of Theorem 1 indicates that with mild restrictions on the cell probabilities, the limiting distribution of the Pearson χ^2 statistic (s observations, n cells) as $n \to \infty$ and $s \to \infty$ is normal if $s^2/n \to \infty$. Further, if the cells are equiprobable, then the limiting distribution is Poisson if $s^2/n \to c > 0$ and degenerate if $s^2/n \to 0$.

38. Tests of Contagion and Time Effect in Accident Proneness. (Preliminary Report.) HOWARD TUCKER, University of California.

This paper considers the case of linear contagion in the theory of accident proneness treated by Miss Grace E. Bates in "Contributions to the theory of Contagion" (to appear in *Ann. of Math. Stat.*). In her paper Miss Bates derived a UMPU test for zero contagion against non-zero contagion in the absence of time effect, based on the exact times at which accidents occur, and in the case of one type of exposure to accidents. In this paper a model is considered for r types of exposure, and the Bates test is extended for this model. For the situation considered by Miss Bates, a test is derived based on the number of accidents which occur in certain time intervals. The power of this test is lower than, but comparable to, that of the Bates test. This test is also extended for the multi-exposure model. Finally, for any fixed value of the contagion parameter, a test is derived for the hypothesis of zero time-effect against non-zero time effect, based on the number of accidents in certain time intervals.

39. Chi Square Test of Goodness of Fit for a Class of Cases of Dependent Observations. JOSEPH PUTTER, University of California.

In the ordinary multinomial situation, we have a sequence of vector r.v. $\{X_n , \cdots , X_{nm}\}$ such that, given $X_{n1} , \cdots , X_{n,k-1}$, the (binomial) r.v. X_{nk} is $\mathcal{B}(n - X_{n1} - \cdots - X_{n,k-1} , p_k)$; we know that $\Sigma_{k=1}^{m}[X_{nk} - (n - X_{n1} - \cdots - X_{n,k-1})p_k]^2 / (n - X_{n1} - \cdots - X_{n,k-1})p_k(1 - p_k)$ is asymptotically distributed as χ_m^2. This is extended to the case where p_k is replaced by $p_{nk}(X_{n1} , \cdots , X_{n,k-1})$, that is, the probability of an observation falling in the kth class depends on the number of observations in the preceding classes. In this case, under some restrictions on the functions p_{nk}, the corresponding statistic is still asymptotically a χ_f^2, but the number of d.o.f. is often reduced. For example, in testing the goodness of fit of a simple chain-binomial model of an epidemic, we get $f = 1$ no matter how many generations we observe.

40. Approximate Probability Values for Observed Number of Successes from Statistically Independent Binomial Events with Unequal Probabilities. JOHN E. WALSH, Lockheed Aircraft Corporation.

Let us consider a number of statistically independent binominal events with possibly unequal probabilities for "success". The principal problem considered is that of determining the probability that the observed number of "successes" equals one of a specified set of values. Given the "success" probabilities, this paper presents an approximate expression which ordinarily furnishes this probability to a reasonable accuracy and is not difficult to evaluate. This approximate expression is obtained by expanding the true expression in terms of the differences between the "success" probabilities and their arithmetic average. If the variation among the event probabilities has known bounds and the upper bound is not too large, usable results often can be obtained in terms of these bounds and the average of the event probabilities. Inverse use of the results for situations of this type sometimes will yield approximate confidence intervals and significance tests for the average of the event probabilities.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Fred C. Andrews, formerly Research Associate in the Applied Mathematics and Statistics Laboratory, Stanford University, is now Assistant Professor of Mathematics and Assistant Statistician in the College of Agriculture at the University of Nebraska.

Dr. R. R. Bahadur is now an Assistant Professor in the Committee on Statistics of the University of Chicago.

T. A. Bancroft, director of the Statistical Laboratory, Iowa State College, has been on a 3-month assignment during fall quarter in the Near East and India at the request of the Food and Agriculture Organization of the United Nations. During the first 6 weeks he visited experiment stations and ministries of agriculture in Egypt, Syria, Iran, and Iraq to collect information on the present use of experimental designs and survey techniques in research investigations. Since then he has given a program of seminars at an Experimental Designs Training Center sponsored by FAO in New Delhi.

Gerald W. Barnes, formerly a graduate student at Indiana University, has accepted an instructorship in Psychology at the University of Maine.

Richard H. Brown has accepted a position of Assistant Professor of Mathematics at Boston University.

Edward C. Bryant returned to his position as Chairman of the Dept. of Statistics, University of Wyoming, last July after a year's study in the Dept. of Statistics, Iowa State College.

Douglas G. Chapman of the Department of Mathematics, University of Washington, has received a Guggenheim Fellowship for research in the statistical theory of population estimation and will be at the University of Oxford during the academic year 1954–55.

Dr. Louis J. Cote has been appointed Assistant Professor of Mathematics at Purdue University. He will be associated with the Statistical Laboratory.

Edwin L. Crow, formerly employed at the U. S. Naval Ordnance Test Station, has accepted a position as Mathematical Statistician with the National Bureau of Standards, Boulder, Colorado.

The Council and Trustees of the American Mathematical Society announce the election of Dr. John H. Curtiss to the position of Executive Director of the Society. Dr. Curtis is a fellow of the Institute of Mathematical Statistics and of the American Statistical Association. He assumed his duties on September 1, 1954. The Headquarters of the Society are at 80 Waterman Street, Providence, Rhode Island.

Joseph F. Daly has been granted a leave by the Bureau of the Census to spend a year at the Statistical Laboratory, Stanford University.

George G. den Broeder, Jr., formerly with the Weapon Systems Analysis

Department of the Northrop Aircraft Company, has accepted an appointment as Research Associate with the Purdue University, Statistical Laboratory.

Gerard T. Dobrindt, formerly a graduate fellow in the Mathematics Department, St. Louis University, received the degree of Master of Science (Research) from that University in June 1954 and is now employed as an Analytical Statistician in the Surveillance Branch, Weapons Systems Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Aberdeen, Maryland.

Dr. David B. Duncan has recently resigned his position as Professor of Statistics at Virginia Polytechnic Institute, Blacksburg, Virginia. He has accepted an appointment as Statistical Consultant in the Statistical Laboratory and Professor of Agronomy (statistics) at the University of Florida, Gainesville, Florida.

A. Ross Eckler received his Ph. D. degree from Princeton University in June 1954 and is a member of the Technical Staff, Bell Telephone Laboratories, Whippany, New Jersey.

W. T. Federer is on sabbatical leave from Cornell University for the period September 1954 to September 1955 and is working with researchers on their experiments, doing routine consulting and solving various statistical problems, at the Experiment Station of the Hawaiian Sugar Planters' Association and the Pineapple Research Institute.

Charles F. Federspiel will spend the academic year 1954–55 doing graduate work at the Institute of Statistics, Raleigh, North Carolina.

John E. Freund is spending this year as a visiting professor in the Department of Statistics, Virginia Polytechnic Institute.

H. H. Goode has been appointed Professor in the Electrical Engineering Department of the University of Michigan in addition to his duties as Director of the Willow Run Research Center and Assistant Director of the Engineering Research Institute.

W. J. Hall, having returned from a year's study in England as a Fulbright Scholar, completed the Ph. D. requirements in statistics at the University of North Carolina in September and is now a member of the technical staff, Bell Telephone Laboratories, New York.

D. G. Hays, candidate for Ph. D. in Sociology at Harvard University, is conducting research on stochastic models for social interaction while a Fellow of the Center for Advanced Study in the Behavioral Sciences, Menlo Park, California.

Roger Heimlich, formerly Research Assistant and Graduate Student, Purdue University, has received a commission in the U. S. Public Health Service and is serving in the Statistics Section, Epidemiology Branch of the Communicable Disease Center, Atlanta, Georgia.

J. Stuart Hunter has obtained his Ph. D. degree in Experimental Statistics, Institute of Statistics, North Carolina State College in June 1954, and is presently employed as an Experimental Statistician by the American Cyanamid Company, New York.

Koichi Ito is spending the 1954–55 academic year at the Institute of Statistics, University of North Carolina after receiving his Ph. D. from St. Louis University.

Dr. C. H. Kraft of Michigan State College has been appointed to an Acting Assistant Professorship at the University of California.

D. V. Lindley is on leave from Cambridge University for the academic year 1954–55 and holds an Assistant Professorship in the Committee on Statistics, University of Chicago for this period.

Dean L. Lindstrom, formerly at the Statistical Laboratory, Purdue University, is now employed as Senior Aerophysics Engineer at the Fort Worth Division of Convair.

Allen L. Mayerson, Principal Actuary of the New York State Insurance Department, has taken a year's leave of absence to study at the University of Paris. Mr. Mayerson has been granted a Fulbright scholarship to study mathematical statistics and applications to actuarial work. His studies will be principally concerned with the Theory of Risk as developed by European Actuaries and Statisticians. Mr. Mayerson has recently been elected a member of the Association des Actuairies Suisses and of the Instituto de Actuarios Espanoles.

Professor Frederick Mosteller, Harvard University, has been awarded a Fund for the Advancement of Education Fellowship for 1954–55 and is visiting the Committee on Statistics, University of Chicago.

Peter G. Moore has returned to a lectureship at University College, London after spending a year at Princeton University as a Commonwealth Fund Fellow.

Mervin Muller has spent the past academic year as a Fellow in Mathematics at the National Bureau of Standards, Institute for Numerical Analysis, University of California, Los Angeles and upon completion of his Ph. D. in August has accepted appointment as Instructor, Mathematics Department, Cornell University.

Sidney I. Neuwirth, formerly affiliated with the Committee on Research of the American Medical Association, Chicago, has accepted the position of Director of the Statistics Division at the Mutual Insurance Advisory Association, New York.

Monroe L. Norden, a Senior Associate of Dunlap and Associates, Inc., Stamford, Connecticut, has been appointed as Engineering Scientist in the Statistical Research and Consulting Group of the Research Division of the College of Engineering, New York University, to work on problems centering about Design of Experiments, Operations Research and the statistical problems that arise in the many engineering problems which involve the representation of physical processes by stochastic processes.

Boyd Z. Palmer, formerly with the American Friends Service Committee, is now employed as a statistician in the Quality Control Department, Timken Roller Bearing Company.

Stanley Reiter, formerly a member of the staff at Stanford University, has been appointed Assistant Professor of Economics and Mathematics at Purdue University.

Donald L. Richter, following a tour of duty with the U. S. Army, has resumed his studies in the Department of Statistics, University of North Carolina.

Robert L. Rogers is now serving as Statistician of Pictsweet Foods, Inc. of Mt. Vernon, Washington, a division of Stokely-Van Camp.

Dr. Jacques St.-Pierre having completed 3 years of graduate work at the Institute of Statistics, University of North Carolina, has returned to Université de Montréal. He will be in charge of the recently organized Center of Statistics.

Marion M. Sandomire left the Atomic Energy Commission, upon the move of its New York Office to Oak Ridge, and has become Staff Statistician at the U. S. Naval Radiological Defense Laboratory in San Francisco.

Dr. Morris Skibinsky, who received his degree in 1954 from the University of North Carolina, Chapel Hill, has been appointed Research Associate in the Statistical Laboratory, Purdue University.

George W. Snedecor will serve as a consultant in experimental statistics for the Institute of Statistics, Consolidated University of North Carolina, for 4 months beginning January 1955.

Richard H. Titman, formerly with the General Electric Company, Richland, Washington, has accepted the position of research Engineer with the Operations Analysis Group of CONVAIR, Division of General Dynamics Corporation, San Diego.

William Van der Bijl, Meteorologist, Research Associate on leave from Royal Netherlands Meteorological Institute, has been granted a Postdoctoral Award for Study in Statistics for the year 1954–55 by the Committee on Statistics of the University of Chicago.

David Lee Wallace, formerly Moore Instructor in Mathematics, Massachusetts Institute of Technology, has been appointed Assistant Professor of Statistics in the Committee on Statistics at the University of Chicago.

Sidney Weiner is employed as Analytical Statistician with Aeronautical Radio Inc., Washington, D. C.

John R. B. Whittlesey, formerly a graduate student at the Institute of Statistics, Chapel Hill, North Carolina, is now a private in the Army, working as an Instrumentation Analyst with the Aberdeen Bombing Mission, Edwards Air Force Base, Edwards, California.

Dr. Max A. Woodbury has been granted leave from the University of Pennsylvania and will serve as Principal Investigator of the Logistics Research Project (ONR), The George Washington University, Washington, D. C. for the coming year.

New Members

The following persons have been elected to membership in the Institute

August 10, 1954 to November 16, 1954

Beckhart, Ann S. (Mrs. G. H.), B.A. (Vassar College), Graduate Student, Rutgers University, *1019 Sycamore Street, Haddon Heights, New Jersey.*

Beckhart, G. H., A.B. (Princeton), Graduate Student, Rutgers University, *1019 Sycamore Street, Haddon Heights, New Jersey.*

Brown, Byron W., Jr., B.A. (Univ. of Minnesota), Instructor, Biostatistics Division, Department of Public Health, University of Minnesota, Minneapolis 14, Minnesota, *Q 179 "U" Village, Minneapolis 14, Minnesota*

Carter, Richard Leston, M.S. (Southern Methodist Univ.), Graduate Student, University of North Carolina, Chapel Hill, North Carolina, *Rt. #3, Box 381, Chapel Hill, North Carolina.*

Chasen, Sylvan H., M.S. (Emory Univ.), Supervisory Mathematician, United States Navy (Naval Air Test Center), Patuxent River, Maryland, *Town Creek Manor, California, Maryland.*

Cox, D. R., Ph.D. (Univ. of Leeds), Assistant Lecturer in Mathematics, Statistical Laboratory, University of Cambridge, St. Andrew's Hill, Cambridge, United Kingdom.

Gupta, Shanti Swarup, M.A. (Delhi Univ.), Research Assistant and Graduate Student, Institute of Statistics, University of North Carolina, Chapel Hill, North Carolina.

Horberg, Arthur, M.S. (DePaul Univ.), Junior Mathematician, Institute for Air Weapons Research, University of Chicago, Museum of Science and Industry, Chicago 37, Illinois, *9640 S. Chappel Avenue, Chicago 17, Illinois.*

Karlin, Samuel, Ph.D. (Princeton Univ.), Associate Professor of Mathematics, California Institute of Technology, Pasadena, California. Temporary, academic year 1954–55: Stanford University, Statistics Department, Stanford, California.

Kawata, Tatsuo, Ph.D. (Tohoku Univ.), Professor of Mathematics, Tokyo Institute of Technology, Meguro-ku Oh-Okayama, Tokyo, Japan.

Lasman, Lonnie L., B.A. (Univ. of Minnesota), Graduate student, University of Minnesota, Minneapolis 14, Minnesota and Research Assistant, Department of Agricultural Economics, Institute of Agriculture, University of Minnesota, St. Paul 8, Minnesota, *1358 Kenneth, St. Paul 2, Minnesota.*

McFeely, Franklin S., M.A. (Bucknell Univ.), Graduate Research Assistant in Statistics, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia.

Maruyama, Gisiro, Ph.D. (Kyusyu Univ.), Professor, Department of Mathematics, Ochanomizu University, 35 Otsuka-machi Bunkyo-ku, Tokyo, Japan.

Murty, V. N., M.A. (Andhra Univ.), Graduate Student, *404 Connor Dormitory, Chapel Hill, North Carolina.*

O'Connor, Aloysius J., Sign writer, Schenectady Sign Service, 834 State Street, Schenectady, New York, *1344 Santa Fe Street, Schenectady, New York.*

Sukhatme, Balkrishna V., M.A. (Delhi Univ.), Research Assistant and Graduate Student, Statistical Laboratory, Department of Mathematics, University of California, Berkeley 4, California.

Switzer, Duane R., B.S. (Case Inst.), Graduate Student, Western Reserve University, Department of Statistics, Dr. W. D. Berretoni, University Circle, Cleveland, Ohio, *5328 Cato Street, Maple Heights, Ohio.*

Udagawa, Masatomo, Ph.D. (Univ. of Tohoku), Assistant Professor, Faculty of Science in Tokyo Educational University, Otsuka Bunkyo-ku, Tokyo, Japan.

Ugaheri, Tadashi, Ph.D. (Tokyo Univ.), Assistant Professor, Tokyo Institute of Technology, O-Okayama Meguro-ku, Tokyo, Japan.

Cooperative Graduate Summer Sessions in Statistics

The University of Florida, North Carolina State College, Virginia Polytechnic Institute and the Southern Regional Education Board are jointly sponsoring a series of cooperative summer sessions in statistics.

The first of these cooperative summer sessions was held during the summer of 1954 at Virginia Polytechnic Institute and the second session will be held at the University of Florida, June 20 to July 29, 1955. A session is scheduled to be held at North Carolina State College in 1956 and another at Virginia Polytechnic Institute in 1957. Each summer session will last six weeks and each course will carry approximately three semester hours of graduate credit.

The faculty for 1955 session at the University of Florida will include: Professor R. L. Anderson, North Carolina State College; Professor D. B. Duncan, University of Florida; Professor Boyd Harshbarger, Virginia Polytechnic Institute; Professor Carl E. Marshall, Oklahoma A. and M. College; Professor Herbert A. Meyer, University of Florida; Professor George E. Nicholson, Jr., University of North Carolina; Professor Phillip J. Rulon, Harvard University; Professor Walter L. Smith, University of North Carolina; and Professor Dudley E. South, University of Florida.

Courses to be offered this summer are: Statistical Methods I, Statistical Methods II (Design of Experiments), Statistical Theory I, Statistical Theory II (Inference and Least Squares), Advanced Analysis I, Theory of Sampling, Theory of Statistical Inference, Mathematics for Statistics, Statistical Research in Education and Psychology and Seminar on Recent Advances in Statistics.

Inquiries should be addressed to:

Professor Herbert A. Meyer
Statistical Laboratory
University of Florida
Gainesville, Florida

Summer Sessions at Berkeley, California

This year's program at the Statistical Laboratory of the University of California, Berkeley, California, consists of two sessions: June 20–July 30 and August 1–September 10, 1955. The faculty of the summer sessions will include Professor G. E. Bates of Mt. Holyoke College, South Hadley, Massachusetts; Professor J. Neyman, Professor Charles H. Kraft and Mr. Howard G. Tucker of the Statistical Laboratory, University of California.

The program includes undergraduate courses primarily meant for students transferring from other centers who would like to embark on advanced studies in Berkeley during the regular academic year. Professor Neyman will be available for consultations on work leading to higher degrees. There will be no graduate course program. However, graduate students may be interested in a series of lectures and seminars to be given through July and August in connection with the second part of the Third Berkeley Symposium on Mathematical Statistics and

Probability. The scholars who promised to participate in this event are: T. W. Anderson, Columbia University, M. S. Bartlett, University of Manchester, J. Berkson, Mayo Clinic, David Blackwell, Howard University and University of California, A. J. L. Blanc-Lapierre, Université d'Alger, J. Doob, University of Illinois, W. Feller, Princeton University, R. Fortet, Institut Henri Poincaré, A Girshick, Stanford University, J. M. Hammersley, Oxford University, J. L. Hodges, Jr., University of California, W. Hoeffding, University of North Carolina, Lucien LeCam, University of California, Erich L. Lehmann, University of California, P. Lévy, l'École Polytechnique, H. Robbins, Columbia University, Herman Rubin, Stanford University, and C. M. Stein, Stanford University.

REPORT OF THE IOWA CITY MEETING OF THE INSTITUTE OF MATHEMATICAL STATISTICS

The sixty-third meeting, a Central Regional meeting, of the Institute of Mathematical Statistics was held at the State University of Iowa, Iowa City, Iowa, on November 26–27, 1954. The meeting was in conjunction with a meeting of the American Mathematical Society. An address entitled *Results of Functions Analytic in a Half Plane and Their Applications to Moments and Distribution Functions* was presented by Professor R. P. Boas, Northwestern University.

The following 29 members of the Institute attended:

F. C. Andrews, I. W. Burr, A. T. Craig, E. L. Crow, J. E. Folkert, W. A. Golomski, L. A. Goodman, J. Gurland, W. W. Gutzman, H. O. Hartley, J. L. Harter, R. V. Hogg, E. H. Jebe, R. J. Jessen, L. Katz, J. H. B. Kemperman, O. Kempthorne, L. A. Knowler, C. F. Kossack, R. M. Kozelka, W. Kruskal, D. V. Lindley, F. W. Lott, Mrs. M. D. Lum, M. R. Mickey, W. C. Ross, Jr., R. Stephens, J. Waksberg, and D. L. Wallace.

The program of the meeting was as follows:

FRIDAY, NOVEMBER 26, 1954

9:00–10:20 a.m. Contributed Papers

Chairman: Leo A. Goodman, University of Chicago

(1) *The Distribution of the Number of Components of a Random Mapping Function, Preliminary Report*, Leo Katz, Michigan State College, and Jay E. Folkert, Michigan State College and Hope College

(2) *Approximate Sequential Tests for Hypotheses about the Proportion of a Normal Population to One Side of a Given Number*, William Kruskal, University of Chicago

(3) *A Note on Some Limit Theorems in Probability Theory*, John Gurland, Iowa State College

(4) *Uniqueness of Latin Square Association Schemes for Partially Balanced Incomplete Block Designs, Preliminary Report* (By title), D. M. Mesner, Purdue University

(5) *On Bounds for the Normal Integral* (By title), J. T. Chu, University of North Carolina

2:00–3:00 p.m. Address

Chairman: JOHN GURLAND, Iowa State College
(1) *Results on Functions Analytic in a Half Plane and Their Applications to Moments and Distribution Functions*, R. P. BOAS, Northwestern University

3:15–5:30 p.m. Sampling Surveys

Chairman: LEO A. GOODMAN, University of Chicago
(1) *Precision of Unbiased Ratio Estimates*, H. O. HARTLEY, Iowa State College
(2) *Some Finite Population Unbiased Ratio and Regression Estimators*, R. M. MICKEY, Iowa State College
(3) *Some Methods of Estimation of Variances for Complex Sample Designs*, J. WAKSBURG, Bureau of the Census

SATURDAY, NOVEMBER 27, 1954

9:30–11:45 a.m. Industrial Statistics

Chairman: IRVING W. BURR, Purdue University
(1) *On Tolerance Limits*, J. H. B. KEMPERMAN, Purdue University
(2) *Contagious Distributions and Their Potential Use in Industry*, LEO KATZ, Michigan State College
(3) *The Location of Optimum Operating Conditions*, DAVID L. WALLACE, University of Chicago

1:00–3:15 p.m. Estimation of Error in Experiments

Chairman: CARL F. KOSSACK, Purdue University
(1) *Linear Models*, O. KEMPTHORNE, Iowa State College
(2) *Interactions with Experimental Units*, M. B. WILK, Iowa State College
(3) *Rules for Determining Error Terms in Hierarchical and Partially Hierarchical Models*, MARY LUM, Wright Air Development Center
Discussion: L. HARTER, Wright Air Development Center

WILLIAM KRUSKAL
Associate Secretary

REPORT OF THE BERKELEY MEETING OF THE INSTITUTE OF MATHEMATICAL STATISTICS

December 27–30

The sixty-fourth meeting of the Institute of Mathematical Statistics and the seventeenth annual meeting was held at the University of California, Berkeley, California, on December 27–30, 1954 in conjunction with the national annual meeting of the American Association for the Advancement of Science. All sessions of the Institute of Mathematical Statistics were joint with Section A of the AAAS. A number of sessions were joint with the American Statistical Association, the American Statistical Association-Committee on Statistics in the Physical Sciences, American Physical Society, Astronomical Society of the Pacific, Biometric Society-Western North American Region, Ecological Society of

America, Philosophy of Science Association, Western Psychological Association, American Association for the Advancement of Science, Section A—Mathematics, Section B—Physics, Section D—Astronomy, Section I—Psychology, Section L—History and Philosophy of Science, Section M—Engineering, Section N—Medical Sciences, Section Q—Education and the Third Berkeley Symposium on Mathematical Statistics and Probability. The following members of the Institute attended:

Israel J. Abrams, Forman S. Acton, Om P. Aggarwal, T. W. Anderson, Kenneth J. Arnold, George A. Baker, Edward W. Barankin, Grace E. Bates, Joseph Berkson, Robert A. Bieber, Allan Birnbaum, Z. W. Birnbaum, David Blackwell, Julius R. Blum, Charles H. Boll, A. H. Bowker, Bernice Brown, H. D. Brunk, Herman Chernoff, C. L. Chiang, John T. Chu, Randolph Church, C. West Churchman, Charles R. Clark, Kermit G. Clemans, Edward P. Coleman, William D. Commins, Arnold Court, L. M. Court, Edwin L. Crow, Joseph F. Daly, Cuthbert Daniel, R. C. Davis, Robert Dorfman, F. W. Dresch, W. J. Dixon, John A. Dudman, Mary Elveback, Benjamin Epstein, James D. Esary, Mark W. Eudey, Edward A. Fay, Thomas Ferguson, Evelyn Fix, Martin Fox, William R. Gaffey, Robert S. Gardner, David W. Gaylor, G. E. Ghormley, Edgar J. Gilbert, E. S. Gordon, Clayborn L. Graves, Harris T. Guard, John Gurland, Harry H. Harman, T. E. Harris, David G. Hays, Warren M. Hirsch, Paul G. Hoel, William C. Hoffman, Joseph L. Hodges, Jr., J. P. Imhof, Stanley Isaacson, R. F. Jarrett, T. A. Jeeves, Walter Jennings, Mark Kac, Williams E. Kane, Samuel Karlin, Leo Katz, H. Konijn, Lambert H. Koopmans, Charles H. Kraft, Martin Karkowski, William Kruskal, Lucien LeCam, Erich L. Lehmann, R. A. Leibler, Roy Leipnik, Gerald J. Lieberman, Julius Lieblein, Richard F. Link, Arvid Lonseth, D. C. Lowry, E. Lukacs, Mary D. Lum, R. K. Maggy, F. Massey, Donald F. Mela, A. M. Mood, Roger A. Moore, L. E. Moses, Frederick Mosteller, J. Neyman, Edwin G. Olds, Donald B. Owen, Toby Oxtoby, Woodrew W. Page, Emanuel Parzen, J. Putter, Robert Putz, K. V. Ramachandran, Bayard Rankin, A. T. Bharucha-Reid, Robert Read, Mina Rees, George J. Resnikoff, Paul R. Rider, William L. Roach, Harry G. Romig, M. Rosenblatt, A. R. Roy, Herman Rubin, David Rubinstein, Jagdish S. Rustagi, Marion M. Sandomire, A. E. Sarhan, Richard Savage, Henry Scheffé, Lorraine Schwartz, Arthur Shapiro, Milton Sobel, Herbert Solomon, Melvin D. Springer, George P. Steck, Charles M. Stein, Margaret D. Stein, David S. Stoller, B. V. Sukhatme, Joseph V. Talacko. William F. Taylor, Henry Teicher, D. Teichroew, A. J. Thomasian, Fred H. Tingey, Donald R. Truax, Howard G. Tucker, William Vanderbyl, Sidney Weiner, Irving Weiss, Oscar Wesler, John R. B. Whittlesey, Alvin D. Wiggins, S. S. Wilks, Max A. Woodbury, J. Yerushalmy, Royal K. Zeigler.

The program follows:

MONDAY, DECEMBER 27, 1954

9:00 a.m. Statistics in Biology and Genetics—With Third Berkeley Symposium, Biometric Society (WNAR), Section A of AAAS, American Statistical Association, and Ecological Society of America

Chairman: Roy E. Clausen, University of California

Papers:
1. *Struggle for Existence*, Thomas Park, University of Chicago; Jerzy Neyman and Elizabeth L. Scott, University of California.
2. *Some Genetic Problems in Controlled Populations*, Everett Dempster, University of California.

3. *Some Genetic Problems in Natural Populations*, James Crow and Motoo Kimura, University of Wisconsin

9:00 a.m. Applications of Poisson and Exponential Distributions to Physics and Industry—With American Statistical Association and Section A of AAAS

Chairman: Stanley Isaacson, Iowa State College
Papers:
1. *Estimates of Bounded Relative Errors for the Poisson and Other Parameters*, Meyer A. Girshick, Stanford University. (Presented by Herman Rubin, Stanford University.)
2. *On Estimating the Mean Life of a Radioactive Source*, Richard F. Link, Sandia Corporation
3. *Survey of Some New Techniques in Lifetesting when the Underlying Distribution is Exponential*, Benjamin Epstein, Wayne University
4. *Inference Problems in Poisson Processes*, Allan Birnbaum, Columbia University

1:30 p.m. Statistical Mechanics—With Third Berkeley Symposium, American Statistical Association, American Physical Society and Sections A and B of AAAS

Chairman: Mina Rees, Hunter College
Papers:
1. *Hopf-Wiener Equation Systems and Matrix Factorization*, Norbert Wiener, Massachusetts Institute of Technology
2. *Foundation of Kinetic Theory of Gases*, Mark Kac, Cornell University
3. *Effect of Dimensionality on Long Range Order in Crystals*, Elliott Montroll, Office of Naval Research

1:30 p.m. Design of Experiments in Fisheries Work—With Biometric Society (WNAR), and Section A of AAAS

Chairman: Richard Van Cleve, University of Washington
Papers:
1. *Biological Assumptions Involved in Estimating Mortalities to Downstream Migrant Salmon Passing Dams*, Charles O. Junge, Jr., Washington State Department of Fisheries
2. *Use of Logbook Data in the Measurement of Distribution and Abundance of Commercial Fish Stocks*, M. B. Schaefer, Inter-American Tropical Tuna Commission
3. *Some Remarks on the Design of a Sampling Program of a Fishery for a Measure of Fishing Intensity*, T. M. Widrig, U. S. Fish and Wildlife Service

4:00 p.m. Special Invited Paper—With Third Berkeley Symposium

Chairman: T. E. Harris, Rand Corporation
Paper: *Large Sample Theory in the Parametric Case*, Herman Chernoff, Stanford University

8:00 p.m. 1954 Council Meeting

President: Edwin G. Olds, Carnegie Institute of Technology

TUESDAY, DECEMBER 28, 1954

9:00 a.m. Statistics in Medicine and Public Health—With Third Berkeley Symposium, American Statistical Association, Biometric Society (WNAR), and Sections A and N of AAAS

Chairman: Charles E. Smith, University of California

Papers:
1. *Problems of Contagion*, William F. Taylor, School of Aviation Medicine
2. *Statistical Problems in Medical Diagnoses*, Chin L. Chiang, J. L. Hodges, Jr. and Jacob Yerushalmy, University of California
3. *Some Statistical Problems Arising from Retrospective Studies*, Jerome Cornfield, National Institute of Health

Discussion: Albert T. Reid, Columbia University, Ahmed E. Sarhan, University of North Carolina

9:00 a.m. Principle of Invariance—With Third Berkeley Symposium and Section A of AAAS

Chairman: Erich L. Lehmann, University of California

Papers:
1. *Canonical Representation of an Invariant Decision Problem*, Meyer A. Girshick, Stanford University. (Presented by Herman Rubin, Stanford University)
2. *Generalized Hunt-Stein Theorem*, Herman Rubin, Stanford University
3. *Admissibility of Invariant Decision Procedures*, Charles M. Stein, Stanford University

1:30 p.m. Invited Address—With Third Berkeley Symposium

Chairman: Henry Scheffé, University of California

Paper: *On Some Problems Arising in the Spectral and Regression Analysis of Time Series*, Murray Rosenblatt, University of Chicago

1:30 p.m. Statistics in Astronomy: The Spatial Distribution of Galaxies—With Third Berkeley Symposium, Astronomical Society of the Pacific, and Sections A and D of AAAS

Chairman: Joel Stebbins, Lick Observatory

Papers:
1. *Statistics of Cluster of Galaxies—Distribution of Centers, Angular Dimensions, Structure, Luminosity Function of Member Galaxies*, Fritz Zwicky, Mount Wilson and Palomar Observatories
2. *Statistics of Images of Galaxies with Particular Reference to Clustering*, Jerzy Neyman and Elizabeth L. Scott, University of California; C. Donald Shane, Lick Observatory
3. *Galaxies, Statistics and Relativity*, George C. McVittie, University of Illinois

2:30 p.m. Invited Papers

Chairman: Edwin G. Olds, Carnegie Institute of Technology

Papers:
1. *Random Variables and the General Theory of Variables*, Karl Menger, Illinois Institute of Technology
2. *Multi-Stage Decision Procedures*, Meyer A. Girshick, Stanford Univer-

sity, Samuel Karlin, California Institute of Technology and Stanford University and H. L. Royden, Stanford University

4:00 p.m. Contributed Papers I

Chairman: Paul G. Hoel, University of California, Los Angeles

Papers:

1. *Maximum Likelihood Estimates of Monotone Parameters*, H. D. Brunk, University of Missouri
2. *Extension of Certain Classes of Contagious Distributions*, John Gurland, Iowa State College
3. *Recurrent Markov Processes II, Preliminary Report*, T. E. Harris, Rand Corporation
4. *The Role of Perks' Distribution in the Theory of Wiener's Stochastic Function*, Joseph Talacko, Marquette University and University of California
5. *On the Orthogonality of Measures and the Existence of Consistent Tests*, Charles Kraft, University of California
6. *The Minimax Character of the Neyman-Pearson Critical Region*, L. M. Court, Diamond Ordnance Fuze Laboratories
7. *On the Existence of Linear Regressions*, T. S. Ferguson, University of California
8. *On the Asymptotic Distribution of U-Statistics Modified by the Introduction of Estimates of Parameters, Preliminary Report*, B. V. Sukhatme, University of California
9. *Essentially Complete Classes of Experiments*, Sylvain Ehrenfeld, New York City. (By title)
10. *A Theorem on Uniform Convergence of Families of Sequences of Random Variables*, Emanuel Parzen, Columbia University. (By title)
11. *Decision Procedures for the Comparison of Exponential Populations*, F. S. McFeely and J. E. Freund, Virginia Polytechnic Institute. (By title)
12. *The Efficiency of some Nonparametric Competitors of the t-Test*, J. L. Hodges, Jr. and E. L. Lehmann, University of California. (By title)

8:00 p.m. AAAS Presidential Address

Address: *A Half Century of Quantum Physics*, E. U. Condon, Director of Research, Corning Glass Works, and Retiring President, AAAS.

WEDNESDAY, DECEMBER 29, 1954

9:00 a.m. Special Invited Paper—With Third Berkeley Symposium

Chairman: David Blackwell, Howard University and University of California

Paper: *Probabilistic Treatment of Phenomena: Stochastic Models and Interpolatory Formulae*, Jerzy Neyman, University of California

10:00 a.m. Probability and Induction—With Third Berkeley Symposium, American Statistical Association, Philosophy of Science Association, and Sections A and L of AAAS

Chairman: William R. Dennes, University of California

Papers:

1. *Problems of Values in Relation to Induction*, C. West Churchman, Case Institute of Technology

2. *The Measurement of Utility and Subjective Probability*, Patrick C. Suppes, Stanford University
3. *Toward an Objectivistic Theory of Probability*, Edward W. Barankin, University of California
4. *Inference from Confidence Limits and from Standard Errors*, Joseph Berkson, Mayo Clinic

1:30 p.m. Comparison of Experiments—With Third Berkeley Symposium and Section A of AAAS

Chairman: Z. William Birnbaum, University of Washington
Papers:
1. *Introduction*, David Blackwell, Howard University and University of California
2. *Comparison of Experiments in the Infinite Case and the Use of Invariance in Determining Sufficiency*, Charles Boll, Hughes Aircraft Company
3. *On the Comparison of Information Provided by Experiments*, Max A. Woodbury, University of Pennsylvania and George Washington University

1:30 p.m. Statistics in Astronomy: The Distribution of Stars in the Hertzsprung-Russell Diagram—With Third Berkeley Symposium, American Statistical Association, Astronomical Society of the Pacific and Sections A and D of AAAS

Chairman: Otto Struve, University of California
Papers:
1. *The H-R Diagram*, Bengt Stromgren, Yerkes Observatory
2. *The Spectra and Other Properties of Stars Lying Below the Normal Main Sequence*, Jesse L. Greenstein, California Institute of Technology
3. *Evidence for Sequences in the Color-Luminosity Relationship for the M-Dwarfs*, Gerald E. Kron, Lick Observatory
4. *Photoelectric Studies of Stellar Magnitudes and Colors*, Harold Johnson, Lowell Observatory
5. *Observed Relationships between Luminosity Color and Mass for Stars in the Solar Neighborhood*, Olin Eggen, Lick Observatory

Discussion: Mary Hanania, University of California

4:30 p.m. Contributed Papers II

Chairman: Leo Katz, Michigan State College
Papers:
1. *Multi-level Continuous Sampling Plans*, Gerald Lieberman, Stanford University, and Herbert Solomon, Teachers College, Columbia University
2. *Numerical Values for Covariances of Order Statistics for Samples of Size Twenty and Less from the Normal Distribution*, D. Teichroew, University of California
3. *Estimation of the Parameters of the Rectangular and the Exponential Populations from Singly and Doubly Censored Samples*, A. E. Sarhan, University of North Carolina.
4. *Estimation from "Censored" Samples of Extreme Data*, Julius Lieblein, National Bureau of Standards
5. *On the Probability Integral Associated with a Certain Multivariate Test*, K. V. Ramachandran, University of North Carolina
6. *Some Uses of Quasi-ranges*, J. T. Chu, University of North Carolina

7. *Inadmissibility of the Usual Estimate for the Mean of a Multivariate Normal Distribution*, Charles M. Stein, Stanford University
8. *A Locally Optimal Test for the Independence of two Poisson Variables*, Mohamed S. Ahmed, University of California.
9. *Identification of a Certain Stochastic Structure*, Henry Teicher, Purdue University
10. *Statistical Estimation of the Endurance Limit*, E. J. Gumbel, Columbia University. (By title)
11. *On the Efficiency of Estimates in Successive Multiphase Sampling (Preliminary Report)*, B. D. Tikkiwal, University of North Carolina. (By title)

7:30 p.m. Business meeting

President: Edwin G. Olds, Carnegie Institute of Technology

9:00 p.m. IMS Beer Party—With Section A, AAAS

THURSDAY, DECEMBER 30, 1954

9:00 a.m. Mathematical Statistics—With Third Berkeley Symposium and Section A of AAAS

Chairman: Wilfred J. Dixon, University of Oregon

Papers:
1. *On the Characterization of Populations by Properties of Suitable Statistics*, Eugene Lukacs, Office of Naval Research
2. *Decision Theory for Polya Type Distributions*, Samuel Karlin, California Institute of Technology and Stanford University

10:00 a.m. Statistics in Industrial Research—With Third Berkeley Symposium, American Statistical Association, and Sections A and M of AAAS

Chairman: Fred L. Hotes, University of California

Papers:
1. *Recent Developments in Sampling Inspection*, Albert H. Bowker, Stanford University
2. *Statistical Problems of Life Testing*, Milton Sobel, Bell Telephone Laboratories
3. *Efficient Experimental Design in Industrial Research*, Cuthbert Daniel, New York City

1:30 p.m. Statistics in Psychology—With Third Berkeley Symposium, American Statistical Association, Western Psychological Association, and Sections A, I, and Q of AAAS

Chairman: Joy P. Guilford, University of Southern California

Papers:
1. *Some Stochastic Models for Learning Theory*, Frederick Mosteller, Harvard University
2. *Statistical Methods in Psychometric Research*, Herbert Solomon, Columbia University
3. *Statistical Inference in Factor Analysis*, Theodore W. Anderson, Columbia University

Discussion: Charles O. Neidt, University of Nebraska

1:30 p.m. The Meaning of Probability to the Engineer, Mathematician and Physicist—With American Statistical Association, and Section A of AAAS

Chairman: H. L. Royden, Stanford University
Papers: Symposium
Discussion:
1. David Blackwell, Howard University and University of California
2. Wendell A. Horning, Ramo Wooldridge Corporation
3. E. T. Jaynes, Stanford University
4. H. L. Royden, Stanford University

4:00 p.m. Contributed Papers III

Chairman: Edward W. Barankin, University of California
Papers:
1. *Detection and Hypothesis Testing by Linear Methods*, Emanuel Parzen, Columbia University
2. *Probability of Indecomposability of a Set Under Random Mappings*, Leo Katz, Michigan State College
3. *On the Near Monotonic Character of the Power Function of a Certain Multivariate Test*, S. N. Roy and K. V. Ramachandran, University of North Carolina
4. *Simultaneous Distribution of the Mean, Standard Deviation, and Range for Probability Density Functions of Doubly Infinite Range, Preliminary Report*, Melvin D. Springer, U. S. Naval Ordnance
5. *A Class of Asymptotic Tests of Composite Hypotheses*, J. Neyman, University of California
6. *The Transitions of a Markovian Process from Set to Set*, Bayard Rankin, Massachusetts Institute of Technology
7. *Confidence Intervals for the Ratio and for the Difference of Two Probabilities, Preliminary Report*, R. R. Bahadur and W. H. Kruskal, University of Chicago
8. *Limit Theorems for Conditional Distributions*, G. P. Steck, University of California
9. *Tests of Contagion and Time Effect in Accident Proneness, Preliminary Report*, Howard Tucker, University of California
10. *Chi Square Test of Goodness of Fit for a Class of Cases of Dependent Observations*, Joseph Putter, University of California
11. *Approximate Probability Values for Observed Number of Successes from Statistically Independent Binomial Events with Unequal Probabilities*, John E. Walsh, Lockheed Aircraft Corporation. (By title)

6:30 p.m. Dinner of Mathematicians and Statisticians—With Third Berkeley Symposium and Section A of AAAS

Speaker: Charles B. Tompkins, University of California, Los Angeles

4:00 p.m. 1955 Council Meeting

President: Henry Scheffé, University of California, Berkeley

T. A. JEEVES
Assistant Secretary

MINUTES OF THE ANNUAL BUSINESS MEETING, 1954

The meeting was called to order by President E. G. Olds at 7:30 P.M. in Room 117 Dwinelle Hall, University of California, Berkeley, California.

The minutes of the business meeting of September 12, 1954 were read by the Secretary. The minutes were approved as read.

Ballots for the election of a President-Elect, for the election of five members of the Council and for voting on an amendment to the Constitution were collected and the tellers retired to determine the results. The results were, after adjournment of the business meeting, announced as follows:

David Blackwell was elected President-Elect for 1955, President for 1956.

R. L. Anderson, Leo Goodman, P. G. Hoel, L. J. Savage and Herbert Solomon were elected to the Council for the term January 1, 1955 to December 31, 1957.

The motion to amend the constitution passed. The second paragraph of Article 9 of the constitution was thereby amended by replacing the phrase, "The publications of the Institute" by "The publication of the official journal." The wording of Article 9 is therefore:

> *The Annals of Mathematical Statistics* shall be the official journal of the Institute. Other publications may be authorized by the Council.
>
> The publication of the official journal shall be supervised by the Editor, with the assistance of the Associate Editors and such committees as the Council may approve.

The Secretary reported the recommendation of the Council that the membership amend the first sentence of the fourth paragraph of Article 1 of the Bylaws by deleting the word "whether" and by deleting the phrase "or to other publications". The Secretary moved the adoption of this amendment. The motion was seconded and carried. The fourth paragraph of Article 1 of the Bylaws now therefore reads:

> The Editor, subject to the direction of the Council, shall have charge of all editorial matters relating to the official journal. He shall, with the advice and consent of the Council, appoint an Editorial Committee of not less than twelve members to cooperate with him for definite terms. All appointments to the Editorial Committee shall terminate with the appointment of a new Editor.

The President submitted his report for the year 1954.[1]

The Secretary submitted his report for the year 1954.[2]

The Treasurer submitted an informal report and announced that a formal report would be submitted after the close of the year.

The Editor submitted his report for the year 1954.[3]

The Program Coordinator submitted an informal report on plans for meetings during 1955.

The incoming President was introduced and took the chair. He said it would

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[1] The report of the President appears on page 179 in this issue of the *Annals*.

[2] The report of the Secretary appears on page 186 in this issue of the *Annals*.

[3] The report of the Editor appears on page 187 in this issue of the *Annals*.

be very difficult to meet the high standards set by his predecessor. He remarked that only the Executive Committee knew how much the retiring President had given of himself. He expressed their appreciation.

President Scheffé called for any new business.

After a very brief discussion of matters related to the business of the Institute, a motion to adjourn was made, seconded and carried and the meeting adjourned at 8:20 P.M.

REPORT OF THE PRESIDENT OF THE INSTITUTE FOR 1954

The names of the members, who have served the Institute in various official capacities during the past year, appear at the close of this report. To all these, and to the many other members who have given their unselfish help, I wish to express my deep appreciation on behalf of the Institute. Furthermore, I offer them my personal thanks for the aid, encouragement and sound advice they have given me.

Because of the efforts of the above-mentioned members the Institute has continued to fulfill its purposes as "a society for encouraging the development, dissemination and application of mathematical statistics", as set forth in its constitution. Our official journal has maintained and enhanced its high reputation. Excellent programs have been presented at our national meeting in Montreal and Berkeley, and at our sectional meetings in Ithaca, Gainesville, Pasadena and Iowa City. Committees have been exploring the opportunities for new activities which promise to be of help to our cause. The funds to support our various services have been carefully nurtured and our sound financial position has been buttressed by prudent investments.

The above statement briefly summarizes the successful work of the Institute for 1954. The remainder of my report will supply more specific information regarding the various contributions which helped to insure this success.

As noted by M. H. Hansen in the President's Report for 1953, the Editor and Secretary-Treasurer bear the brunt of the work of the Institute. While the Associate Editors and Cooperating Members of the Editorial Board richly deserve recognition, it must be clear to all that the major responsibility for the *Annals* belongs to E. L. Lehmann himself. In addition to this constant load, the Editor has served as an official member of several committees.

K. J. Arnold carried two burdens, as the secretary and also as the treasurer; either one of these would have strained an ordinary person. Also, he has been a member of six committees, and as chairman of the Committee to Prepare a Manual for the Guidance of Officers he has made a signal contribution to the efficient operation of the Institute. Through his efforts the officers of the Institute have been supplied with a codification of Council actions and a description of the functions of standing committees, together with a list of past officers, Rietz lecturers and other official representatives, in a form for ready reference.

This has been a year when the members of the Council should be publicly com-

mended for an unusual quantity and quality of labor on behalf of the Institute. In addition to their deliberation, at three formal meetings, Council members have been called upon to discuss and settle many matters by correspondence. Their prompt, yet careful, consideration of problems, which at times demanded almost daily attention, should not go unmarked.

Leo Katz, the Program Coordinator, has been a member of all of the program committees, besides serving as Chairman of the Program Committee for the annual meeting. His unselfish contribution of time and energy deserves special recognition.

D. B. DeLury, R. A. Bradley, D. A. Darling and T. E. Harris carried program responsibilities as chairmen for the Montreal meeting, and Eastern, Central and Western regional meetings respectively. Elizabeth Scott was vice-chairman for the meeting at Berkeley. For the success of the meetings in their respective areas, Associate Secretaries Lionel Weiss, William Kruskal and Evelyn Fix deserve much credit. Furthermore, the work of the various assistant Secretaries should not be overlooked.

Discussion of meetings should include mention of the work of Leo Goodman's Committee in arranging for special invited addresses and of the gratitude of the Institute to the men who delivered the addresses. The list for 1954 was as follows:

Summer Meeting: W. Hoeffding, "Non-Parametric Large Sample Theory"
Annual Meeting: H. Chernoff, "Large Sample Theory in the Parametric Case"
Annual Meeting: J. Neyman, "Probabilistic Treatment of Phenomena: Stochastic Models and Interpolatory Formulae."
(Arrangements for the last-mentioned address were made by the 1953 Committee.)

The steady growth of the Institute is fostered by the work of the promotional committees. For his Committee on Individual Memberships, W. D. Baten reports the addition of 62 new members.

Upon recommendation of the Rietz Lecture Committee, the Council approved the appointment of Will Feller as the next Rietz Lecturer. His lecture is planned for 1955.

B. F. Kimball and his Committee on Professional Standards of Statisticians in Government Service are making a survey of required qualifications, salaries, and duties for this group. A comprehensive report is contemplated.

The Editorial Committee for the Publication of Wald's Selected Papers, under the chairmanship of T. W. Anderson, Jr. has completed its work and early publication by the McGraw-Hill Company has been scheduled.

Late in 1953, J. Neyman agreed to take the chairmanship for a Committee to plan for a summer statistical institute. The report of this Committee was received and approved by the Council at the Montreal meeting. Following a recommendation made in the report, a Committee was appointed to prepare detailed plans for a statistical institute to be held in the summer of 1955 at Berkeley. This Committee, consisting of D. H. Blackwell, T. E. Harris, and H. E. Robbins, probably will be ready to report to the membership in the near future.

In contrast to its condition in the lean years of youth, the Institute now finds

itself with a modest surplus for investment. The Treasurer, with the advice of Mortimer Spiegelman's Finance Committee, has been making an effort to find sound and profitable repositories for all funds not needed for immediate expenses.

Late in 1953, a Committee on Activities and Development was appointed with A. H. Bowker as Chairman. This Committee has proposed that the Institute cooperate in the publication of a series of statistical research monographs.

Some of our members have expressed the wish that the scheduling of the annual meeting between Christmas and New Year's could be avoided. C. C. Craig and his Committee to Explore the Desirability of Changing the Time of Winter Meetings are preparing a questionnaire on the subject for distribution to a sample of our membership. It might be mentioned that our desire to hold joint meetings with both mathematical and other statistical organizations severely restricts the independence of our actions.

Early this year, the Secretary requested the appointment of a Committee to advise him on the adequacy of physical facilities for meetings. This Committee, with J. L. Hodges, Jr., as Chairman, has served its purpose well.

Among the many duties of the incoming President, Henry Scheffé, has been his service as Chairman of a Committee to Re-examine the Constitution and By-Laws. Manifestations of the considerable work which has been accomplished probably will reach the membership in 1955.

A Committee consisting of T. W. Anderson, Jr., Chairman; H. Cramér, W. Feller, M. A. Girshick, and S. S. Wilks, was appointed to nominate the Editor of the Annals for the term of 1956–58. The election of the new editor by the Council must take place well in advance of the beginning of his term of office so that he and his staff may begin work on manuscripts. I am very happy to report that the committee was able to persuade the present editor, E. L. Lehmann, to accept nomination for another term. He was duly re-elected by the Council at the Montreal meeting.

P. S. Dwyer, Chairman; W. E. Deming, and Churchill Eisenhart were appointed as members of a Committee to nominate a Secretary and a Treasurer to succeed Kenneth J. Arnold, whose term as Secretary-Treasurer expires June 30, 1955. Over the years, the work of the combined office has so increased that it now seems necessary to take advantage of the constitutional provision which permits separate officers. Upon nomination by Professor Dwyer's Committee, the 1954 Council, at its Berkeley meeting, elected George E. Nicholson, Jr., as Secretary and Albert H. Bowker as Treasurer. The terms for both officers are for the period, July 1, 1955 to June 30, 1958.

As usual, our official representatives have been effective. Harold Hotelling is completing his term in the Council of the American Association for the Advancement of Science. S. S. Wilks has been appointed to a further three-year term on the Division of Mathematics, National Research Council, by President Bronk of the National Academy of Sciences. W. G. Madow and T. W. Anderson, Jr., are completing another year of helpful service on the Inter-Society Committee on the

Mathematical Training of Social Scientists. It should be noted that Professor Madow accepted the responsibility for organizing this successful Committee and has served as its chairman from the beginning.

J. C. R. Li was asked to represent the Institute at the inauguration of Owen Meredith Wilson as President of the University of Oregon.

The following members served as official delegates from the Institute to the International Congress of Mathematicians at Amsterdam:

D. H. Blackwell
J. Neyman
H. E. Robbins
C. Stein
S. S. Wilks

Upon nomination by the Committee on Fellows, the Council elected the following persons as Fellows of the Institute:

R. R. Bahadur
H. E. Daniels
D. A. Darling
D. A. S. Fraser
L. M. LeCam

The following persons have been appointed to serve on the Nominating Committee for 1955:

W. J. Dixon, Chairman
M. A. Girshick
M. G. Kendall
H. B. Mann
H. Nisselson
A. W. Tucker

Committees of the Institute, 1954

I. The Council and Committees of the Council.

(a) Elected members of the Council

Term expires 1954
A. H. Bowker
T. C. Koopmans
H. E. Robbins
H. G. Romig

Term expires 1955
W. G. Cochran
Churchill Eisenhart
Henry Scheffé
J. W. Tukey

Term expires 1956
T. W. Anderson, Jr.
Joseph Berkson
Z. W. Birnbaum
D. H. Blackwell
W. G. Madow

(b) Executive Committee

E. G. Olds, *President*
Henry Scheffé, *President-Elect*
K. J. Arnold, *Secretary-Treasurer*
E. L. Lehmann, *Editor*

(c) Committee on Fellows

Term expires 1954	Term expires 1955
Gerhard Tintner	W. G. Madow
S. S. Wilks, *Chairman*	Edward Paulson

Term expires 1956
D. H. Blackwell
Howard Levene

(d) Associate Secretaries
Evelyn Fix, Western Region
W. H. Kruskal, Central Region
Lionel Weiss, Eastern Region

(e) Associate Treasurer
E. S. Pearson

(f) Associate Editors

D. H. Blackwell	Wassily Hoeffding
H. E. Daniels	W. G. Madow
J. L. Hodges, Jr.	A. M. Mood

J. Wolfowitz

II. Editorial Committee
(a) E. L. Lehmann, *Editor, Chairman*
(b) Associate Editors listed immediately above
(c) Cooperating members

Z. W. Birnbaum	H. B. Mann
R. C. Bose	G. E. Noether
Herman Chernoff	Edward Paulson
K. L. Chung	M. P. Peisakoff
J. F. Daly	H. E. Robbins
D. A. Darling	S. N. Roy
J. L. Doob	L. J. Savage
T. E. Harris	Herbert Solomon
P. G. Hoel	C. M. Stein
J. Kiefer	Lionel Weiss
W. H. Kruskal	M. A. Woodbury
Soloman Kullback	

III. Committees Related to Program
(a) Annual Meeting—Berkeley

Leo Katz, *Chairman*	D. G. Chapman
Elizabeth L. Scott, V. Chr.	M. A. Girshick
Evelyn Fix, Assoc. Secy.	T. E. Harris
J. A. Jeeves, Asst. Secy.	G. B. Price
Z. W. Birnbaum	Mina Rees
D. H. Blackwell	F. F. Stephan

(b) September Meeting—Montreal

D. B. DeLury, *Chairman*	G. J. Lieberman
Lionel Weiss, Assoc. Secy.	W. E. Patte
Roger Lessard, Asst. Secy.	Howard Raiffa
R. L. Anderson	R. G. D. Steel
I. W. Burr	D. J. Thompson
	D. F. Votaw

(c) Eastern **Region**

R. A. Bradley, *Chairman*
Lionel Weiss, Assoc. Secy.
I. Blumen, Asst. Secy.
H. A. Meyer, Asst. Secy.
I. J. D. Bross
Glenn Burrows
Max Halperin
J. E. Jackson
J. Kiefer
Gottfried Noether

(d) Central Region

D. A. Darling, *Chairman*
W. H. Kruskal, Assoc. Secy.
A. T. Craig, Asst. Secy.
John Gurland
A. W. Kimball
Ingram Olkin
Murray Rosenblatt
W. F. Taylor
Henry Teicher

(e) Western Region

T. E. Harris, *Chairman*
L. A. Aroian, Asst. Secy.
W. J. Dixon
J. L. Hodges, Jr.
S. W. Nash
H. Rubin
D. Teichroew

(f) Program Coordinator (Ex-officio member of all Program Committees)
Leo Katz

(g) Special Invited Papers

L. A. Goodman, *Chairman*
T. W. Anderson, Jr.
D. H. Blackwell
R. A. Bradley
D. A. Darling
D. B. DeLury
T. E. Harris
Leo Katz
E. L. Lehmann (Ex-Officio)
P. R. Rider

IV. Promotional Committees

(a) Individual Memberships

W. D. Baten, *Chairman*
Evelyn Fix
E. L. Kaplan
E. S. Keeping
O. E. Lancaster
M. R. Mickey, Jr.
E. W. Pike
M. E. Wescott
Sigmund Zobel

(b) Academic Institutional Memberships

T. A. Bancroft, *Chairman*
A. C. Cohen, Jr.
G. B. Dantzig
P. C. Hammer
Boyd Harshbarger
L. A. Knowler
E. B. Mode
F. M. Weida

(c) Non-Academic Institutional Membership

H. G. Romig, *Chairman*
C. A. Bennett
M. A. Brumbaugh
A. V. Feigenbaum
R. P. Gage
F. J. Halton, Jr.
E. P. King
C. E. Noble
P. S. Olmstead

V. Other Committees

(a) Nominating Committee

Appointed by 1953 President M. H. Hansen

Max Woodbury, *Chairman*
Walter Bartky
R. A. Bradley
Edward Paulson
George Kuznets

(b) Rietz Lecture Committee
J. Neyman, *Chairman*
Harald Cramér
Harold Hotelling

(c) Professional Standards of Statisticians in Government Service
B. F. Kimball, *Chairman*
K. A. Brownlee
R. W. Burgess
Besse B. Day
Churchill Eisenhart
G. M. Harrington
P. G. Homeyer
A. S. Householder
George Kuznets
Joseph Lev
Herbert Marhsall
R. E. Patton
J. E. Walsh

(d) Editorial Committee for the publication of Wald's Selected Papers
T. W. Anderson, Jr., *Chairman*
Harald Cramér
H. A. Freeman
E. J. Lehmann
J. L. Hodges, Jr.
A. M. Mood
C. M. Stein

(e) Planning Committee for Summer Statistical Institute
J. Neyman, *Chairman*
T. W. Anderson, Jr.
Z. W. Birnbaum
D. H. Blackwell
J. L. Doob
W. Feller
J. Gurland
W. G. Madow
F. C. Mosteller
H. A. Meyer
G. E. Nicholson
M. Rosenblatt
M. Sobel
C. M. Stein
R. M. Thrall

(f) Committee to Prepare Manual for Guidance of Officers
K. J. Arnold, *Chairman*
P. S. Dwyer
M. A. Girshick
M. H. Hansen
E. G. Olds
Henry Scheffé

(g) Advisory Committee on Statistical Computations
Z. W. Birnbaum, *Chairman*
A. H. Bowker
Churchill Eisenhart
C. C. Craig

(h) Committee on Exchanges
P. S. Dwyer, *Chairman*
K. J. Arnold (*Ex-Officio*)
E. L. Lehmann (*Ex-Officio*)

(i) Finance Committee
Mortimer Spiegelman, *Chairman*
C. H. Fischer
K. J. Arnold

(j) Committee on Activities and Development
A. H. Bowker, *Chairman*
R. C. Bose
G. W. Brown
J. H. Curtiss
W. E. Deming
K. W. Halbert
H. O. Hartley
E. L. Lehmann
E. S. Marks
Frank Riordan
H. E. Robbins

(k) Committee on Committee Procedures
H. E. Robbins, *Chairman*
K. J. Arnold
Z. W. Birnbaum
J. H. Curtiss
H. W. Norton

(l) Committee to Explore the Desirability of Changing Time of Winter Meetings
C. C. Craig, *Chairman*
R. L. Anderson
J. L. Hodges, Jr.
C. F. Kossack
R. B. Murphy

(m) Advisory Committee on Physical Facilities for Meetings
J. L. Hodges, Jr., *Chairman*
Leo Katz (*Program Co-ordinator*)
G. E. Nicholson

(n) Committee to Re-examine the Constitution and By-Laws
Henry Scheffé, *Chairman*
T. W. Anderson, Jr.
K. J. Arnold (*Ex-Officio*)
W. G. Cochran
Arnold Court
T. C. Koopmans
S. S. Wilks

(o) Committee to Nominate an Editor
T. W. Anderson, Jr., *Chairman*
H. Cramér
W. Feller
M. A. Girshick
S. S. Wilks

(p) Committee to Nominate a Secretary and a Treasurer
P. S. Dwyer, *Chairman*
W. E. Deming
C. Eisenhart

(q) Committee to Organize and Conduct the 1955 Summer Statistical Institute
D. H. Blackwell, *Chairman*
J. E. Harris
H. E. Robbins

Representatives of the Institute for 1954

To the American Association for the Advancement of Science
Harold Hotelling (Term expires December 31, 1954)
To the National Research Council, Division of Mathematics
S. S. Wilks (Term expires June 30, 1957)
To the Policy Committee for Mathematics
Joseph F. Daly (Term expires December 31, 1954)
On the Committee on Mathematical Training of Social Scientists
W. G. Madow and T. W. Anderson, Jr.
On the Committee on Establishing a Classification List of Specialties in Mathematics for the National Science Foundation
Benjamin J. Tepping

December 29, 1954

Edwin G. Olds
President

REPORT OF THE SECRETARY-TREASURER OF THE INSTITUTE FOR 1954

The individual membership in the Institute increased from 1399 at the end of 1953 to 1512 at the end of 1954, a net increase of 113 individual members. During the year there was one death, 17 resignations, 40 members dropped for nonpayment of dues, 18 reinstated on payment of dues for previous years and 163 new members accepted. The work of the Membership Committee, in inviting individuals presumably interested in mathematical statistics to apply for membership in the Institute, was responsible for a large share of the new members. Of the

1512 individual members at the end of 1954, 442 were in the special categories set up by the bylaws: 232 were members whose permanent residence is outside the United States and Canada, 185 were student members, 12 were nominees of institutional members, 8 were husband-wife members, 3 were retired from active service and 2 were on active military duty.

The number of institutional members increased from 9 at the end of 1953 to 12 at the end of 1954 with the enrollment of the Statistical Laboratory of the State College of Iowa, the Department of Mathematics of Michigan State College, and the State University of Iowa. Our institutional members are very much appreciated both because of their financial support of the expending program of the Institute and because of the approval of the activities and aspirations of the Institute, which the officers of the Institute believe is implied by this financial support.

During 1954 the Institute held its 59th through 64th meetings. These meetings have been reported in the *Annals* except for the last two, reports of which will appear in the same issue of the *Annals* as will this report. Business meetings were held during the 62nd (16th Summer) meeting and the 64th (17th Annual) meeting. The report of the first of these appeared in the December *Annals* and the report of the second will appear in the same issue of the *Annals* as will this report. The Program Committees are to be congratulated on the excellent programs which have been arranged under the immediate direction of R. A. Bradley, T. E. Harris, D. B. DeLury, Leo Katz and Elizabeth Scott with the overall guidance of our Program Coordinator Leo Katz. The full membership of the Program Committees is given in the President's report. The Assistant Secretaries Isadore Blumen, H. A. Meyer, L. A. Aroian, Roger Lessard, A. T. Craig and T. A. Jeeves are to be congratulated on the physical arrangements and the Associate Secretaries Lionel Weiss, W. H. Kruskal and Evelyn Fix on their performance of the duties of the Secretary with respect to these meetings.

In November a directory was issued with information obtained from members up to October 15, 1954. The information given is more complete than has been available in directories of the Institute for many years.

It has been usual in the past to have the reports of the Secretary and of the Treasurer submitted at the same time. This year the Treasurer suggested and the Executive Committee agreed that the auditor's report should be transmitted to the membership and that the Treasurer's report should be delayed until these two reports could be submitted together. The Treasurer's report for 1954 should therefore appear in the June, 1955 issue of the *Annals*.

K. J. ARNOLD
Secretary

REPORT OF THE EDITOR OF THE ANNALS FOR 1954

The 1954 volume of the *Annals* contained 53 papers and 23 notes. Together with the usual reports, abstracts, and news and notices this brought the total number of pages printed to 826. While this is an increase of more than 100 pages

over the previous volume, the number of papers submitted during the year showed a decrease for the first time in many years.

The special invited paper published this year was by J. Wolfowitz: "Estimation by the Minimum Distance Method in Nonparametric Stochastic Difference Equations".

On behalf of the Editorial board, the Editor takes the opportunity to acknowledge the generous refereeing assistance of the following: G. E. Albert, T. W. Anderson, F. J. Anscombe, R. E. Bechhofer, Joseph Berkson, Max A. Bershad, Julius Blum, G. E. P. Box, A. H. Bowker, A. C. Cohen, W. C. Connor, D. R. Cox, C. C. Craig, Monroe Donsker, Meyer Dwass, B. Epstein, D. A. S. Fraser, M. A. Girshick, Leo Goodman, Ulf Grenander, John Gurland, J. M. Hammersley, Morris H. Hansen, L. H. Herbach, William N. Hurwitz, T. A. Jeeves, Leo Katz, D. G. Kendall, L. LeCam, Eugene Lukacs, F. J. Massey, Lincoln Moses, F. Mosteller, W. L. Nicholson, E. S. Pearson, C. R. Rao, Murray Rosenblatt, I. R. Savage, Henry Scheffé, Elizabeth Scott, Monroe G. Sirken, Rosedith Sitgreaves, Milton Sobel, George Steck, and W. F. Taylor.

The Editor is greatly indebted to Arnold Court and Mrs. Joanne Hand for preparing the manuscripts for the printer and providing other editorial assistance.

E. L. LEHMANN
Editor

December 29, 1954.

PUBLICATIONS RECEIVED

GNEDENKO, B. V. AND A. N. KOLMOGOROV, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Co., Inc., Cambridge, 1954, ix + 264 pp., $7.50.

Naval Research Logistics Quarterly, Vol. 1, No. 1, March, (1954), Office of Naval Research, U. S. Government Printing Office, 74 pp.

JENSEN, ARNE, *A Distribution Model*, Munksgaard, Copenhagen, 1954, 99 pp.

KLEIN, LAWRENCE R., GEORGE KATONA, JOHN B. LANSING AND JAMES N. MORGAN, *Contributions of Survey Methods to Economics*, Columbia University Press, New York, 1954, 269 pp., $5.00.

MASUYAMA, M., *Graphical Method of Statistical Inference*, Maruzen Co., Ltd., Tokyo, 83 pp., $2.50.

KENNEY, J. F. AND E. S. KEEPING, *Mathematics of Statistics*, Part I, D. Van Nostrand Company, Inc., Toronto, New York, London, Third Edition, 1954, xii + 346 pp.

ABDEL-ATY, S. H., "Tables of Generalized k-Statistics", Vol. 41, parts 1 and 2, Biometrika, Statistical Table No. XIX, Cambridge University Press, 1954, 8 pp., 2 shillings.

Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues, edited by Olga Taussky, National Bureau of Standards Applied Mathematics Series 39, 139 pp., $2.00. (Order from the Government Printing Office, Washington 25, D. C.)

KOYCK, L. M., *Distributed Lags and Investment Analysis*, North-Holland Publishing Co., Amsterdam, 1954, v + 111 pp., $2.75.

ESTADISTICA

Journal of the Inter American Statistical Institute

Volume XIII, No. 46 **March 1955**

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Necesidades Inmediatas y Mediatas en Estadísticas Vitales y Sanitarias en Países de Distinto Nivel de Desarrollo ..... HERNÁN ROMERO Y JERJES VILDÓSOLA

The Use of Hospital Statistics to Study Morbidity in the Community
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Utilización de los Censos para Probar la Integralidad del Registro de Nacimientos
ROQUE GARCÍA-FRIAS

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THE AMERICAN STATISTICAL ASSOCIATION

announces the publication of two new monographs:

Statistical Problems of the Kinsey Report

by COCHRAN, MOSTELLER and TUKEY. The evaluation of the statistical methodology used by Kinsey and his associates in their first volume. This study was requested by the Committee for Research in the Problems of Sex of the National Research Council, which is sponsoring Dr. Kinsey's work. The contents include Statistical Problems of the Kinsey Report, Discussion of comments by Selected Technical Reviewers, Comparison with Other Studies, Proposed Further Work, Probability Sampling Considerations, The Interview and The Office, Desirable Accuracy, Principles of Sampling.

The monograph contains 331 pages, plus a foreword, preface and index; bound in blue buckram; $3.00 to ASA members; $5.00 to others.

Proceedings of the Business and Economics Statistics Section.

The papers given at the sessions sponsored by the Business and Economics Statistics Section at the Annual Meeting of the American Statistical Association in Montreal in September 1954. This volume contains papers on Pension Funds, Business Outlook, International Payments, Consumer Survey Data, Forecasting, Employment and Unemployment Statistics, Stock Market, Government Statistics, Measurement of Saving and Investment, Mobilization, Productivity.

Approximately 250 pages, paper bound. Price $2.00 to ASA members; $3.00 to others.

Copies may be ordered directly from the AMERICAN STATISTICAL ASSOCIATION, 1108 Sixteenth Street, N. W., Washington 6, D. C.

ECONOMETRICA

Journal of the Econometric Society

Contents of Vol. 23, No. 1 - January, 1955

W. HAMBURGER:.....The Relation of Consumption to Wealth and the Wage Rate

F. V. WAUGH AND G. L. BURROWS:.....A Short Cut to Linear Programming

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